CSCE 411 – HW4 solution

Exercise 24-3

Description
Arbitrage is the use of discrepancies in currency exchange rates to transform one unit of a currency into more than one unit of the same currency. For example, suppose that 1 U.S. dollar buys 49 Indian rupees, 1 Indian rupee buys 2 Japanese yen, and 1 Japanese yen buys 0.0107 U.S. dollars. Then, by converting currencies, a trader can start with 1 U.S. dollar and buy $49 \times 20 \times 0.0107 = 10.486$ U.S. dollars, thus turning a profit of 4.86 percent.

Suppose that we are given \( n \) currencies \( c_1, c_2, \ldots, c_n \) and an \( n \times n \) table \( R \) of exchange rates, such that one unit of currency \( c_i \) buys \( R[i, j] \) units of currency \( c_j \).

(a). Give an efficient algorithm to determine whether or not there exists a sequence of currencies \(< c_{i_1}, c_{i_2}, \ldots, c_{i_k} >\) such that \( R[i_1, i_2] \cdot R[i_2, i_3] \cdots R[i_{k-1}, i_k] \cdot R[i_k, i_1] > 1 \). Analyze the running time of your algorithm.

(b). Give an efficient algorithm to print out such a sequence if one exists. Analyze the running time of your algorithm.

Solution:

(a). If there is a sequence \(< c_{i_1}, c_{i_2}, \ldots, c_{i_k} >\) such that \( R[i_1, i_2] \cdot R[i_2, i_3] \cdots R[i_{k-1}, i_k] \cdot R[i_k, i_1] > 1 \), then we have

\[
-\sum_{j=1}^{k-1} \log R[i_j, i_{j+1}] - \log R[i_k, i_1] < 0.
\]

We create a directed complete graph \( G = (V, E) \), where each currency \( c_i \) corresponds to a vertex \( v_i \) of \( G \). If \( R[i, j] > 0 \), let the weight of this edge \( w(i, j) \) be \(-\log R[i, j] \); otherwise, let the \( w(i, j) = 0 \). Run Bellman-Ford algorithm starting from an arbitrary vertex on this graph. If there is a negative cycle, then there exists such a sequence. The time complexity is \( O(|V||E|) \).

(b). To find such a sequence. We first create a graph \( G = (V, E) \) as part (a). Then we relax all the edges \( |V| - 1 \) times, as in the Bellman-Ford algorithm. Then we record all of the \( d \) values of the vertices. Then we relax all the edges \( |V| \) more times. Then we check which vertices had their \( d \) values decrease since we record them. All of these vertices must lie on some set of negative weight cycles. Call \( S \) this set of vertices. Run DFS on the induced graph by \( S \) to find a cycle. The time complexity is \( O(|V||E|) \).

Exercise 29.1-5

Description
Convert the following linear program into slack form:

\[
\text{maximize } 2x_1 - 6x_3 \\
\text{subject to: } x_1 + x_2 - x_3 \leq 7 \\
3x_1 - x_2 \geq 8 \\
-x_1 + 2x_2 + 2x_3 \geq 0 \\
x_1, x_2, x_3 \geq 0
\]

What are the basic and nonbasic variables?

Solution:
Step 1: Convert the linear program into standard form by multiple -1 to both sides of second and third constraint.

\[
\text{maximize } 2x_1 - 6x_3 \\
\text{subject to: } x_1 + x_2 - x_3 \leq 7 \\
-3x_1 + x_2 \leq -8 \\
x_1 - 2x_2 - 2x_3 \leq 0 \\
x_1, x_2, x_3 \geq 0 
\]

Step 2: Convert the linear program into slack form:

\[
\begin{align*}
    z &= 2x_1 - 6x_3 \\
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -8 + 3x_1 - x_2 \\
x_6 &= -x_1 + 2x_2 + 2x_3 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{align*}
\]

Thus the basic variables are \((x_4, x_5, x_6)\) and the nonbasic variables are \((x_1, x_2, x_3)\).

**Exercise 29.3-5**

**Description**

Solve the linear program (LP) using SIMPLEX algorithm:

\[
\text{maximize } 18x_1 + 12.5x_2 \\
\text{subject to: } x_1 + x_2 \leq 20 \\
x_1 \leq 12 \\
x_2 \leq 16 \\
x_1, x_2 \geq 0
\]

**Solution.**

Step 1: we convert the linear programming into slack form:

\[
\begin{align*}
    z &= 18x_1 + 12.5x_2 \\
x_3 &= 20 - x_1 - x_2 \\
x_4 &= 12 - x_1 \\
x_5 &= 16 - x_2 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{align*}
\]

The basic solution is \((x_1, x_2, x_3, x_4, x_5) = (0, 0, 20, 12, 6)\) and its objective value is \(z = 0\). We choose to increase the value of \(x_1\).
Step 2:

\[ z = 216 - 18x_4 + 12.5x_2 \]
\[ x_1 = 12 - x_4 \]
\[ x_3 = 8 - x_2 + x_4 \]
\[ x_5 = 16 - x_2 \]
\[ x_1, x_2, x_3, x_4, x_5 \geq 0 \]

The basic solution is \((x_1, x_2, x_3, x_4, x_5) = (12, 0, 8, 0, 16)\) and its objective value is \(z = 216\). We choose to increase the value of \(x_2\).

Step 3:

\[ z = 316 - 5.5x_4 - 12.5x_3 \]
\[ x_1 = 12 - x_4 \]
\[ x_2 = 8 + x_4 - x_3 \]
\[ x_5 = 8 - x_4 + x_3 \]
\[ x_1, x_2, x_3, x_4, x_5 \geq 0 \]

The basic solution is \((x_1, x_2, x_3, x_4, x_5) = (12, 8, 0, 0, 8)\) and its objective value is \(z = 316\). Now, all coefficients in the objective function are negative. Thus the final solution is \(316\).

**Exercise 29.3-6**

**Description**

Solve the following linear program using SIMPLEX:

\[
\text{maximize } 5x_1 - 3x_2 \\
\text{subject to: } x_1 - x_2 \leq 1 \\
2x_1 + x_2 \leq 2 \\
x_1, x_2 \geq 0
\]

**Solution.**

Step 1: we convert the linear programming into slack form:

\[ z = 5x_1 - 3x_2 \]
\[ x_3 = 1 - x_1 + x_2 \]
\[ x_4 = 2 - 2x_1 - x_2 \]
\[ x_1, x_2, x_3, x_4 \geq 0 \]

The basic solution is \((x_1, x_2, x_3, x_4) = (0, 0, 1, 2)\) and its objective value is \(z = 0\). We choose to increase the value of \(x_1\).
Step 2:

\[ z = 5 + 2x_2 - 5x_3 \]
\[ x_1 = 1 - x_3 + x_2 \]
\[ x_4 = -3x_2 + 2x_3 \]
\[ x_1, x_2, x_3, x_4 \geq 0 \]

The basic solution is \((x_1, x_2, x_3, x_4) = (1, 0, 0, 0)\) and its objective value is \(z = 5\). We choose to increase the value \(x_2\).

Step 3:

\[ z = 5 - \frac{11x_3}{3} - \frac{2x_4}{3} \]
\[ x_1 = 1 - \frac{x_2 + x_4}{3} \]
\[ x_2 = \frac{2x_3 - x_4}{3} \]
\[ x_1, x_2, x_3, x_4 \geq 0 \]

The basic solution is \((x_1, x_2, x_3, x_4) = (1, 0, 0, 0)\) and its objective value is \(z = 5\). Now, all coefficients in the objective function are negative. Thus the final solution is 5.