Problem 23.1-3 (25 points):
Show that if an edge \((u, v)\) is contained in some minimum spanning tree \(T\), then it is a light edge crossing some cut of the graph.

Solution:
If we remove edge \((u, v)\) from tree \(T\), we will get a forest of two trees, \(T_1\) and \(T_2\). Let's say \(S\) is the partition of vertices \(V\) containing the vertex \(u\). Then \((S, V-S)\) represents the cut. Suppose \((u, v)\) is not the light edge for the cut \((S, V-S)\) and which is not part of the tree \(T\). Replacing \((u, v)\) in \(T\) by edge \((x, y)\) we can form a new tree \(T'\). Since \(w(u,v) > w(x,y)\), tree \(T'\) is lighter than \(T\). But as per our assumption, \(T\) has to be the MST. Hence, by contradiction, we can say that edge \((u, v)\) crossing some cut of the graph is indeed light edge.

Problem 23.1-5 (25 points):
Let \(e\) be a maximum-weight edge on some cycle of connected graph \(G(V,E)\). Prove that there is a minimum spanning tree of \(G'(V,E-\{e\})\) that is also a minimum spanning tree of \(G\). That is, there is a minimum spanning tree of \(G\) that does not include \(e\).

Solution:
Let's say \(T\) is MST of \(G'\). \(T\) will be spanning tree of \(G'\) as well as both \(G\) and \(G'\) contain same vertices. Now, suppose \(T\) isn't MST of \(G\), then there exists another tree \(T'\) with weight less than \(T\) which is MST of \(G\). Note that, \(T'\) contains edge \(e\) as well. We can simply replace edge \(e\) with any edge \(e'\), not currently part of \(T'\) and having weight less than that of edge \(e\) (as \(e\) has maximum weight), to obtain another tree \(T''\). Now, it very clear that weight of \(T''\) has to be less than that of \(T'\) making it MST of \(G\). But as per our assumption, \(T'\) is the MST of \(G\). This violates our supposition that \(T'\) is the MST of \(G\) and hence \(T=T''\) (not containing edge \(e\)) is indeed the MST of \(G\).

Problem 29.1-4(25 points):
Convert the following linear program into standard form:

\[
\begin{align*}
\text{minimize} & \quad 2x_1 + 7x_2 + x_3 \\
\text{subject to} & \quad x_1 - x_3 = 7 \\
& \quad 3x_1 + x_2 \geq 24 \\
& \quad x_2 \geq 0 \\
& \quad x_3 \leq 0 
\end{align*}
\]

Solution:
Step 1: Convert the minimization objective function to equivalent maximization function by negating all its coefficients:
maximize \ -2x_1 - 7x_2 - x_3 \\
subject to 
\ x_1 - x_3 = 7 \\
\ 3x_1 + x_2 \geq 24 \\
\ x_2 \geq 0 \\
\ x_3 \leq 0 \\

Step 2: Variable \ x_3 \) doesn’t satisfy the non-negativity constraint. Replacing \ x_3 \) with another variable \ x_3' \) such that \ x_3' = -x_3. \\

maximize \ -2x_1 - 7x_2 + x_3' \\
subject to 
\ x_1 + x_3' = 7 \\
\ 3x_1 + x_2 > 24 \\
\ x_2 \geq 0 \\
\ x_3' \geq 0 \\

Step 3: Now, \ x_1 \) also doesn’t satisfy the non-negativity constraint. Replacing \ x_1 \) as \ x_1' - x_1''. \\

maximize \ -2x_1' + 2x_1'' - 7x_2 + x_3' \\
subject to 
\ x_1' - x_1'' + x_3' = 7 \\
\ 3x_1' - 3x_1'' + x_2 \geq 24 \\
\ x_1', x_1'', x_2, x_3 \geq 0 \\

Step 4: Changing the equality constraint by writing two inequality constraints. \\

maximize \ -2x_1' + 2x_1'' - 7x_2 + x_3' \\
subject to 
\ x_1' - x_1'' + x_3' \leq 7 \\
\ x_1' - x_1'' + x_3' \geq 7 \\
\ 3x_1' - 3x_1'' + x_2 \geq 24 \\
\ x_1', x_1'', x_2, x_3 \geq 0 \\

Step 5: Lastly, convert all the \ \geq \) constraints into \ \leq \) constraints by multiplying the inequalities by -1. \\

maximize \ -2x_1' + 2x_1'' - 7x_2 + x_3' \\
subject to 
\ x_1' - x_1'' + x_3' \leq 7 \\
\ -x_1' + x_1'' - x_3' \leq -7
Step 6: For simplicity, replace $x_1'$ with $x_1$, $x_1''$ with $x_4$, $x_3'$ with $x_3$. We have standard form as:

\[
\begin{align*}
\text{maximize} \quad & 2x_1 + 2x_4 - 7x_2 + x_3 \\
\text{subject to} \quad & x_1 - x_4 + x_3 \leq 7 \\
& -x_1 + x_4 - x_3 \leq -7 \\
& -3x_1 + 3x_4 - x_2 \leq -24 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

**Problem 29.1-5 (25 points):**
Convert the following linear program into slack form:

maximize \quad 2x_1 - 6x_3 \\
subject to \quad x_1 + x_2 - x_3 \leq 7 \\
& 3x_1 - x_2 \geq 8 \\
& -x_1 + 2x_2 + 2x_3 \geq 0 \\
& x_1, x_2, x_3 \geq 0

What are the basic and nonbasic variables?

**Solution:**
Step 1: First lets convert $\geq$ inequalities to $\leq$ inequalities multiplying them by -1.

maximize \quad 2x_1 - 6x_3 \\
subject to \quad x_1 + x_2 - x_3 \leq 7 \\
& -3x_1 + x_2 \leq -8 \\
& x_1 - 2x_2 - 2x_3 \leq 0 \\
& x_1, x_2, x_3 \geq 0

Step 2: We have standard form LP. To convert this to slack form LP, lets add three slack variables $x_4$, $x_5$ and $x_6$ for each of the constraints,

\[
\begin{align*}
\text{maximize} \quad & 2x_1 - 6x_3 \\
\text{subject to} \quad & x_4 = 7 - x_1 - x_2 + x_3 \\
& x_5 = -8 + 3x_1 - x_2 \\
& x_4, x_5, x_6 \geq 0
\end{align*}
\]
\[ x_6 = -x_1 + 2x_2 + 2x_3 \]
\[ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \]

Basic variables: \( x_4, x_5, x_6 \).
Non-basic variables: \( x_1, x_2, x_3 \).