Problem 29.3-5 (25 points):
Solve the following linear program using SIMPLEX:
maximize $18x_1 + 12.5x_2$
subject to
$x_1 + x_2 \leq 20$
$x_1 \leq 12$
$x_2 \leq 16$
$x_1, x_2 \geq 0$

Solution:
i) First lets convert the LP into slack form:

maximize $18x_1 + 12.5x_2$
subject to
$x_3 = 20 - x_1 - x_2$
$x_4 = 12 - x_1$
$x_5 = 16 - x_2$
$x_1, x_2, x_3, x_4, x_5 \geq 0$
Our initial basic solution is thus, $x_1 = x_2 = 0$, $x_3=20$, $x_4 = 12$ and $x_5=16$ with value of objective function 0.

ii) Now we can see that we can maximize our objective function by increasing $x_1$ and $x_2$. Lets first choose $x_1$. We can see that to satisfy the non-negativity constraints, the second constraint is tightest one and limits the value of $x_1$. Switching roles of $x_1$ and $x_4$, the LP is:

maximize $216 - 18x_4 + 12.5x_2$
subject to
$x_3 = 8 + x_4 - x_2$
$x_1 = 12 + x_4$
$x_5 = 16 - x_2$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

iii) We now try to maximize our objective function by increasing $x_2$. The tightest constraint limiting the value of $x_2$ is $x_3$. We pivot $x_2$ to $x_3$, the LP now becomes:

maximize $316 - 5.5x_4 + 12.5x_3$
subject to
\[ x_2 = 8 + x_3 - x_3 \]
\[ x_1 = 12 + x_4 \]
\[ x_5 = 8 - x_4 + x_3 \]
\[ x_1, x_2, x_3, x_4, x_5 \geq 0 \]

iv) Now since we don’t have any non basic variables in the objective function with positive coefficients. Our solution is \((x_1, x_2, x_3, x_4, x_5) = (12, 8, 0, 0, 8)\) and objective function thus has maximum value of 316. For our original standard form LP, solution will be \(x_1=12\) and \(x_2=8\). We can check the maximum value is actually 316.

**Problem 29.3-6 (25 points):**
Solve the following linear program using SIMPLEX:

maximize \(5x_1 - 3x_2\)

subject to
\[ x_1 - x_2 \leq 1 \]
\[ 2x_1 + x_2 \leq 2 \]
\[ x_1, x_2 \geq 0 \]

**Solution:**

i) First lets convert the LP into slack form:

maximize \(5x_1 - 3x_2\)

subject to
\[ x_3 = 1 - x_1 + x_2 \]
\[ x_4 = 2 - 2x_1 - x_2. \]
\[ x_1, x_2, x_3, x_4 \geq 0 \]

ii) As only positive coefficient variable in objective function is \(x_1\), we pivot \(x_1\) to \(x_3\). We have:

maximize \(5 - 5x_3 + 2x_2\)
\[ x_1 = 1 - x_3 + x_2 \]
\[ x_4 = 2x_3 - 3x_2 \]
\[ x_1, x_2, x_3, x_4 \geq 0 \]

iii) Now in the above LP, we have \(x_2\) with positive coefficient in the objective function. The last constraint limits the value of \(x_2\) to max 0. We will pivot \(x_2\) to \(x_4\).
Maximize $5 - \frac{11}{3} x_3 - \frac{1}{3} x_4$
$x_1 = 1 - \frac{1}{3} x_3 - \frac{1}{5} x_4$
$x_2 = \frac{1}{5} x_3 - \frac{1}{6} x_4$
$x_1, x_2, x_3, x_4 \geq 0$

iii) Now since we don’t have any non basic variables in the objective function with positive coefficients. Our solution is $(x_1, x_2, x_3, x_4) = (1,0,0,0)$ and objective function thus has maximum value of 5. For our original standard form LP, solution will be $x_1=1$ and $x_2=0$. We can check the maximum value is actually 5.

**Problem 29.3-7 (25 points):**
Solve the following linear program using SIMPLEX:
minimize $x_1 + x_2 + x_3$
subject to
$2x_1 + 7.5x_2 + 3x_3 \leq 10000$
$20x_1 + 5x_2 + 10x_3 \leq 30000$
$x_1, x_2, x_3 \geq 0$

**Solution:**
i) First, we convert this equation to the slack form. We will maximize the negative of objective function:

maximize $z = -x_1 - x_2 - x_3$
subject to
$x_4 = -10000 + 2x_1 + 7.5x_2 + 3x_3$
$x_5 = -30000 + 20x_1 + 5x_2 + 10x_3$
$x_1, x_2, x_3, x_4, x_5 \geq 0$

ii) Now, we note that the initial basic solution is not feasible, because it would leave $x_4$ and $x_5$ being negative. We will convert the above LP into auxiliary LP as:

maximize $z = -x_6$
subject to
$x_4 = -10000 + 2x_1 + 7.5x_2 + 3x_3 + x_6$
$x_5 = -30000 + 20x_1 + 5x_2 + 10x_3 + x_6$
$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

iii) We pivot $x_5$ to $x_6$ as, as it is the basic variable whose value in the basic solution is most negative. After pivoting, we have the LP as:
maximize \( z = -30000 + 20x1 + 5x2 + 10x3 - x5 \)
subject to,
\[
\begin{align*}
x6 &= 30000 - 20x1 - 5x2 - 10x3 + x5 \\
x4 &= 20000 - 18x1 + 2.5x2 - 7x3 + x5 \\
x1, x2, x3, x4, x5, x6 &\geq 0
\end{align*}
\]

iii) Pivoting \( x6 \) to \( x2 \). This gives
maximize \( z = -x6 \)
Subject to
\[
\begin{align*}
x2 &= 6000 - 4x1 - 2x3 + (1/5)x5 - (1/5)x6 \\
x4 &= 35000 - 28x1 - 12x3 + (3/2)x5 - (1/2)x6 \\
x6, x1, x2, x3, x4, x5 &\geq 0
\end{align*}
\]

iv) This slack form is the final solution to the auxiliary problem. Since this solution has \( x6 = 0 \), we know that our initial problem was feasible. Since \( x6 = 0 \), we can just remove it from the set of constraints. We then restore the original objective function, This yields
\[
\begin{align*}
z &= -6000 + 3x1 + x3 - (1/5)x5 \\
x2 &= 6000 - 4x1 - 2x3 + (1/5)x5 \\
x4 &= 35000 - 28x1 - 12x3 + (3/2)x5 \\
x1, x2, x3, x4, x5 &\geq 0
\end{align*}
\]

v) This slack form has a feasible basic solution, and we can now solve it using Simplex.
We choose \( x1 \) as our entering variable. This gives
Maximize
subject
\[
\begin{align*}
z &= -2250 - (2/7)x3 - (3/28)x4 - (11/280)x5 \\
x1 &= 1250 - (3/7)x3 - (1/28)x4 + (3/56)x5 \\
x2 &= 1000 - (2/7)x3 + (1/7)x4 - (1/70)x5 \\
x1, x2, x3, x4, x5 &\geq 0
\end{align*}
\]
At this point, all coefficients in the objective function are negative, so the basic solution is an optimal solution. This solution is \((x1, x2, x3) = (1250, 1000, 0)\) with objective value of function being 2250.

**Problem 29.4-1 (25 points)**
Formulate the dual of the linear program given in Exercise 29.3-5.

**Solution:**
To form the dual, we change the maximization to a minimization, exchange the roles of the right-hand sides and the objective-function coefficients, and replace the less-than-or-equal-to by
a greater-than-or-equal-to. Each of the $m$ constraints in the primal has an associated variable $y_i$ in the dual, and each of the $n$ constraints in the dual has an associated variable $x_j$ in the primal.

We get the dual as:

minimize $20y_1 + 12y_2 + 16y_3$
subject to
$y_1 + y_2 \geq 18$
$y_1 + y_3 \geq 12.5$
$y_1, y_2, y_3 \geq 0$