Problem 1. (35 points)
There are $n$ toys in a toy store, which we call $T_1, \ldots, T_n$. For $i = 1, 2, \ldots, n$, the toy $T_i$ has a price of $p_i$ dollars, where $p_i$ is a positive integer. A child walks into the store with $D$ dollars in his pocket. He would like to spend as much as he can on toys, without exceeding the limit of $D$ dollars.

Design an algorithm to help the child achieve his goal. (The objective value you want to maximize is the amount of money he spends on toys.) Prove the correctness of your algorithm, and analyze its time complexity.

Solution:
We need to use a dynamic programming with the following recursive relation.

$$ S(D, n) = \max \{ S(D - p_n, n - 1) + p_n, \quad \text{if } T_n \text{ is picked}, $$

$$ S(D, n - 1), \quad \text{if } T_n \text{ is not picked}, $$

where $S(D, n)$ is the maximum amount of money he can spend on toys $T_1, \ldots, T_n$ with $D$ dollars. If we pick a toy $T_n$, then the subproblem is how much he can spend on toys $T_1, \ldots, T_n$ with $(D - p_n)$ dollars. If $T_n$ is not picked, then we move on to the case how much he can spend on toys $T_1, \ldots, T_n$ with $D$ dollars. Its time complexity is $O(nD)$.

Problem 2. (30 points)
Let $G = (V, E)$ be a weighted undirected graph, where every edge $eeE$ has a weight $w_e$. All the edge weights in $G$ are unique, namely, no two edges have the same weight. Is it true that $G$ has a unique minimum-spanning tree (namely, $G$ does not have two different minimum-spanning trees)? Prove your conclusion.

Solution:
Suppose that $T_1$ and $T_2$ are both MSTs of $G$ and not identical. Pick an edge $(u, v) \in T_1 - (T_1 \cap T_2)$ such that $w(u, v)$ is smallest. Then, there must be a path $u \sim v$ in $T_2$, and we pick an edge $(x, y) \in T_2$ in the path such that $w(x, y) > w(u, v)$. Here, we observe that $(u, v)$ and the path $u \sim v$ together make a cycle. By choosing $u \sim v$ instead of $(x, y)$ in $T_2$, we can have a new spanning tree $T_2'$ such that $w(T_2') < w(T_2)$, which is a contradiction. Therefore, there is an unique MST.
Problem 3. (35 points)
Solve the following linear program:
\[
\begin{align*}
\text{max} & \quad -x_1 + 3x_2 + x_3 \\
\text{s.t.} & \quad 3x_1 - x_2 + 2x_3 \leq 7 \\
& \quad -2x_1 + 4x_2 \leq 12 \\
& \quad -4x_1 + 3x_2 + 8x_3 \leq 10 \\
& \quad x_1, x_2, x_3 \geq 0 
\end{align*}
\]

Solution:
Convert the linear programming into slack form
\[
\begin{align*}
z &= -x_1 + 3x_2 + x_3 \\
x_4 &= 7 - 3x_1 + x_2 - 2x_3 \\
x_5 &= 12 + 2x_1 - 4x_2 \\
x_6 &= 10 + 4x_1 - 3x_2 - 8x_3 
\end{align*}
\]
The basic solution is \((x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 7, 12, 10)\) and its objective value \(z = 0\). We choose to increase value of \(x_2\) and we have:

Step 1:
\[
\begin{align*}
z &= 9 + \frac{1}{2}x_1 + x_3 - \frac{3}{4}x_5 \\
x_2 &= 3 + \frac{1}{2}x_1 - \frac{1}{4}x_5 \\
x_4 &= 10 - \frac{5}{2}x_1 - 2x_3 - \frac{1}{4}x_5 \\
x_6 &= 1 + \frac{5}{2}x_1 - 8x_3 + \frac{3}{4}x_5 
\end{align*}
\]
The basic solution is \((x_1, x_2, x_3, x_4, x_5, x_6) = (0, 3, 0, 10, 0, 1)\) and its objective value \(z = 9\). We choose to increase value of \(x_1\) and we have:

Step 2:
\[
\begin{align*}
z &= 11 + \frac{3}{5}x_3 - \frac{1}{5}x_4 - \frac{4}{5}x_5 \\
x_1 &= 4 - \frac{4}{5}x_3 - \frac{2}{5}x_4 - \frac{1}{10}x_5 \\
x_2 &= 5 - \frac{2}{5}x_3 - \frac{1}{5}x_4 - \frac{3}{10}x_5 \\
x_6 &= 11 - 10x_3 - x_4 + \frac{1}{2}x_5 
\end{align*}
\]
The basic solution is \((x_1, x_2, x_3, x_4, x_5, x_6) = (4, 5, 0, 0, 0, 11)\) and its objective value is \(z = 11\). We choose to increase \(x_3\) and we have:

Step 3:
The basic solution is \((x_1, x_2, x_3, x_4, x_5, x_6) = (78/25, 114/25, 11/10, 0, 0, 0)\) and its objective value \(z = \frac{583}{50} = 11.66\).