Problem Set 10

Problem 1 (20 points, Problem 34.5-1). The subgraph-isomorphism problem takes two undirected graphs $G_1$ and $G_2$, and it asks whether $G_1$ is isomorphic to a subgraph of $G_2$. Show that the subgraph-isomorphism problem is NP-complete.

Solution. First it’s in NP, where the certificate is just the injective mapping from $G_1$ into $G_2$ so that $G_1$ is isomorphic to its image.

Second, we can do a reduction from clique. Let $G_1$ be a clique of size $k$. The problem asks if $G_2$ has a clique of size $k$. If we could solve the subgraph isomorphism problem quickly, we could solve the clique problem quickly.

Problem 2 (20 points, Problem 34.5-2). Given an integer $m \times n$ matrix $A$ and an integer $m$-vector $b$, the 0-1 integer programming problem asks whether there exists an integer $n$-vector $x$ with elements in the set $\{0,1\}$ such that $Ax \leq b$. Prove that 0-1 integer programming is NP-complete. (Hint: Reduce from 3-CNF-SAT.)

Solution. First, we prove it’s in NP. A certificate could be the $n$-vector $x$, and we can verify in polynomial time to see whether $Ax \leq b$.

Second, we can do a reduction from 3-CNF-SAT. Let $\phi$ be 3-CNF formula with $n$ input variables and $k$ clauses. We construct an instance of 0-1LP as follows: let $A$ be a $(k+2n) \times 2n$ matrix. For $1 \leq i \leq k$, set $A(i,j) = -1$ if $1 \leq j \leq n$ and clause $C_i$ contains the literal $x_j$, and 0 otherwise. For $k+1 \leq i \leq k+n$, set $A(i,j) = -1$ if clause $C_i$ contains the literal $\bar{x}_j$, and 0 otherwise.

For $k+n+1 \leq i \leq k+2n$, set $A(i,j) = -1$ if $i-n-k = j$ or $i-k = j-n$, and 0 otherwise. Let $b$ be a $(k+2n)$-vector. Set the first $k$ entries to -1, the next $n$ entries to 1 and the last $n$ entries to -1. It’s obvious that we construct $A$ and $b$ in polynomial time.

We now show that $\phi$ has a satisfying assignment if and only if there exists a 0-1 vector $x$ such that $Ax \leq b$. First, assume that $\phi$ has a satisfying assignment. For $1 \leq i \leq n$, if $x_i$ is true, make $x[i] = 1$ and $x[n+i] = 0$; otherwise, make $x[i] = 0, x[i+n] = 1$. It’s easy to check that $Ax \leq b$.

Next, we show that any 0-1 solution to $Ax \leq b$ provides a satisfying assignment. Let $x$ be such a 0-1 solution. The last $2n$ inequalities makes that one of $x[i]$ and $x[i+n]$ is 0 and the other is 1 for $1 \leq i \leq n$. Since row $i$ of $b$ is -1 for $1 \leq i \leq k$, it guarantees at least one literal is true in each clause. This makes $\phi$ satisfied.