Problem Set 11

Problem 1 (20 points, Problem 34.5-7). The longest-simple-cycle problem is the problem of determining a simple cycle (no repeated vertices) of maximum length in a graph. Formulate a related decision problem, and show that the decision problem is NP-complete.

Solution. The related decision problem is as follows: given a graph $G$ and an integer $k$, decide if there is a simple cycle of length at least $k$ in the graph $G$. To see that this problem is in NP, just let the certificate be the cycle itself. It is easy to walk along this cycle, and make sure no vertices repeated.

To see that it is NP-hard, we will do a reduction to Hamilton cycle. Suppose we have a graph $G$ and want to know if it is Hamilton. We can create an instance of the decision problem as follows: let $G' = G$ and $k$ be the vertices of $G$. It’s easy to see that $G$ has a Hamiltonian cycle if and only if $G'$ has a simple cycle of length at least $k$.

Problem 2 (20 points, Problem 35.2-3). Consider the following closest-point heuristic for building an approximate traveling-salesman tour whose cost function satisfies the triangle inequality. Begin with a trivial cycle consisting of a single arbitrary chosen vertex. At each step, identify the vertex $u$ that is not on the cycle but whose distance to any vertex on the cycle is minimum. Suppose that the vertex on the cycle that is nearest $u$ is vertex $v$. Extend the cycle to include $u$ by inserting $u$ just after $v$. Repeat until all vertices are on the cycle. Prove that this heuristic returns a tour whose total cost is not more than twice the cost of an optimal tour.

Solution. From the chapter on minimum spanning trees, recall Prim’s algorithm. That’s a minimum spanning tree can be found by repeatedly finding the nearest vertex to the vertices already considered, and adding it to the tree, being adjacent to the vertex among the already considered vertices that is closest to it. The closest-point heuristic for building an approximate traveling-salesman tour is exactly the preorder traversal of a tree built by the Prim’s algorithm. It was shown in this chapter that such a Hamiltonian cycle is a 2 approximation for the cheapest cycle under the given assumption that the weights satisfy the triangle inequality.

Problem 3 (20 points, Problem 35.4-3). In the MAX-CUT problem, we are given an unweighted undirected graph $G = (V, E)$. We define a cut $(S, V - S)$ as in Chapter 23 and the weight of a cut as the number of edges crossing the cut. The goal is to find a cut of maximum weight. Suppose that for each vertex $v$, we randomly and independently place $v$ in $S$ with probability 1/2 and in $V - S$ with probability 1/2. Show that this algorithm is a randomized 2-approximation algorithm.

Solution. For each edge, the probability that this edge crosses the cut is 1/2. Therefore, by linearity of expectation, we know that the expected value of edges that cross the cut is $\frac{|E|}{2}$. We know that the number of the maximum cut is
bounded by $|E|$. Thus this algorithm is a randomized 2-approximation algorithm.