Mid-Term Exam 2

Problem 1 (50 points). Solve the linear program (LP) using SIMPLEX algorithm:

\[
\begin{align*}
\text{minimize} & \quad -4x_1 - x_2 \\
\text{subject to:} & \quad -x_1 + 2x_2 \leq 4 \\
& \quad 2x_1 + 3x_2 \leq 12 \\
& \quad x_1 - x_2 \leq 3 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Solution. Step 1: we convert the linear programming into slack form:

\[
\begin{align*}
z & = 4x_1 + x_2 \\
x_3 & = 4 + x_1 - 2x_2 \\
x_4 & = 12 - 2x_1 - 3x_2 \\
x_5 & = 3 + x_2 - x_1 \\
x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{align*}
\]

The basic solution is \((\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5) = (0, 0, 4, 12, 3)\) and its objective value is \(z = 0\). We choose to increase the value of \(x_1\).

Step 2:

\[
\begin{align*}
z & = 12 + 5x_2 - 4x_5 \\
x_1 & = 3 + x_2 - x_5 \\
x_3 & = 7 - x_2 - x_5 \\
x_4 & = 6 - 5x_2 + 2x_5 \\
x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{align*}
\]

The basic solution is \((\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5) = (3, 0, 7, 6, 0)\) and its objective value is \(z = 12\). We choose to increase the value \(x_2\).

Step 3:

\[
\begin{align*}
z & = 18 - 2x_5 - x_4 \\
x_1 & = \frac{21}{5} - \frac{3x_5}{5} - \frac{x_4}{5} \\
x_2 & = \frac{6}{5} + \frac{2x_5}{5} - \frac{x_4}{5} \\
x_3 & = \frac{29}{5} - \frac{7x_5}{5} + \frac{x_4}{5} \\
x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{align*}
\]

The basic solution is \((\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5) = (21/5, 6/5, 29/5, 0, 0)\) and its objective value is \(z = 18\). Now, all coefficients in the objective function are negative. Thus the final solution is -18.
Problem 2 (50 points). In Olympics 2016, there were athletes from \( n \) countries participating in \( m \) games. Let \( C = \{c_1, c_2, \ldots, c_n\} \) denote the \( n \) countries, and let \( H = \{h_1, h_2, \ldots, h_m\} \) denote the \( m \) games. For simplicity, assume that every country sent at most one athlete to each game. For \( i = 1, 2, \ldots, m \), let \( A_i \subseteq C \) denote the set of countries who sent athletes to participated in the game \( h_i \). Every game produced three medals: the gold medal, the silver medal, and the bronze medal.

For \( j = 1, 2, \ldots, n \), let \( x_j \) denote the total number of medals won by country \( c_j \) after all games ended. Clearly, the vector \((x_1, x_2, \ldots, x_n)\) (which is basically the Medal List of Countries) needs to satisfy certain constraints and therefore cannot be arbitrary. For example, the summation of \( x_1, x_2, \ldots, x_n \) has to be \( 3m \).

The question we consider is this: given any vector \( X = (x_1, x_2, \ldots, x_n) \), is \( X \) a possible outcome of Olympics 2016? Design an efficient algorithm to determine if the answer is “yes” or “no”, prove its correctness, and analyze its time complexity.

Solution. We can create a network as follows: let \( V = \{s, t\} \cup H \cup C \). For each \( c_i \in C \), add a directed edge from \( s \) to \( c_i \) with capacity \( x_i \). For each \( h_i \in H \), add a directed edge from \( h_i \) to \( t \) with capacity 3. For each \( c_i \in C, h_j \in H \), if \( c_i \) sent an athlete to \( h_j \), add a directed edge from \( c_i \) to \( h_j \) with capacity 1.

Calculate the maximum flow of the above network. If the value is \( 3m \) and \( \sum_{i=1}^{n} x_i = 3m \), then \( X \) is a possible outcome. Otherwise, it’s not.