

Let m and D be integers such that $m \leq D$. A *block* is a set of m cells whose levels are from the set $[1, D]$ and are all distinct. Let (c_1, c_2, \dots, c_m) denote those m cell levels. Then by definition, $c_i \in [1, D]$ for $i \in [1, m]$ and $\forall i \neq j, c_i \neq c_j$. For convenience, we call (c_1, c_2, \dots, c_m) a *block*, too, and call $\mathcal{I}(c_1, c_2, \dots, c_m)$ the induced *permutation*. (\mathcal{I} is as defined in the previous section.) If a block B induces a permutation P , then B is called a *realization* of P . Note that a permutation may have multiple realizations. For example, if $m = 6$ and $P = (1, 4, 3, 2)$, then both $(1, 6, 4, 3)$ and $(2, 5, 4, 3)$ are realizations of P .

Let (c_1, c_2, \dots, c_n) be the levels of n cells. Let $v < m$ be an integer and for convenience, let $(n-v)/(m-v)$ be an integer as well. For $i = 1, 2, \dots, \frac{n-v}{m-v}$, let B_i denote the block $(c_{(i-1)(m-v)+1}, c_{(i-1)(m-v)+2}, \dots, c_{(i-1)(m-v)+m})$. Note that the last v cell levels of B_i are also the first v cell levels of B_{i+1} , so we say these two blocks overlap by v . We say (c_1, c_2, \dots, c_n) , or $(B_1, B_2, \dots, B_{(n-v)/(m-v)})$, is a *cell-level sequence* consisting of blocks that overlap by v . For $i = 1, 2, \dots, \frac{n-v}{m-v}$, let the m levels in B_i be all distinct and $P_i = \mathcal{I}(B_i)$. Then the sequence induces $\frac{n-v}{m-v}$ permutations $(P_1, P_2, \dots, P_{(n-v)/(m-v)})$, called the induced *permutation sequence*. And we call $(B_1, B_2, \dots, B_{(n-v)/(m-v)})$ its *realization*. Again, a permutation sequence may have multiple realizations.

Definition 1 (BOUNDED RANK MODULATION $\mathcal{C}(n, m, D, v)$) In a bounded rank modulation (BRM) code $\mathcal{C}(n, m, D, v)$, every codeword is a permutation sequence $(P_1, P_2, \dots, P_{(n-v)/(m-v)})$ that has at least one realization. Let $|\mathcal{C}(n, m, D, v)|$ denote the number of codewords in code \mathcal{C} . Then, the capacity of the code is

$$\text{cap}(\mathcal{C}) = \lim_{n \rightarrow \infty} \frac{\log |\mathcal{C}(n, m, D, v)|}{n}.$$

In general, allowing overlap between permutations can increase capacity. When there is no overlap (i.e., $v = 0$), the BRM code has capacity $\frac{\log m!}{m}$. When $v > 0$, the capacity may increase because every permutation consumes just $m - v$ cells on average.

III. BRM CODE WITH ONE OVERLAP AND CONSECUTIVE LEVELS

In this section, we study a special BRM code that allows efficient computation of its capacity. First, we present a computational method based on constrained systems. Detailed definitions are shown in [11].

Since $c_i \in [1, D]$ for $i \in [1, m]$, the BRM code is a *constrained system*. Let $G = (V, E, L)$ be a deterministic labeled graph representing $\mathcal{C}(n, m, D, v)$, where V, E and L are the state set, the edge set, and the edge labeling, respectively. $L(u, v) = l$ is denoted by $u \xrightarrow{l} v$, $l \in S_m$ (the symmetric group). If A_1, A_2, \dots, A_k are,

the *adjacency matrices* of the irreducible components in G , then

$$\text{cap}(\mathcal{C}(n, m, D, v)) = \frac{\max_{1 \leq i \leq k} \log \lambda(A_i)}{m - v} \quad (1)$$

where $\lambda(A)$ is largest positive eigenvalue of A [9].

Example 2 A BRM code $\mathcal{C}(n, 2, 3, 1)$ can be represented by the deterministic graph G in Figure 1 (a). Each state represents the current cell level. $S_2 = \{12, 21\}$, $V = \{1, 2, 3\}$, and $E = \{(i, i+1) | i = 1, 2\} \cup \{(i, i-1) | i = 2, 3\}$. The labeling is $L(i, i+1) = 12, \forall i = 1, 2$ and $L(i, i-1) = 21, \forall i = 2, 3$. For example, the path along the states 1, 2, 3, and 2 is a realization of the permutation sequence $(12, 12, 21)$. The adjacency matrix of G is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

By (1), the capacity is $\log(\lambda(A)) = 0.5$.

Notice in Example 2, every block $B_i = (c_i, c_{i+1})$ consists of two consecutive integers, i.e., $|c_i - c_{i+1}| = 1$. If we generalize this idea to arbitrary $D \geq 2$ but keep $m = 2$, and $v = 1$, we get the constrained system in Figure 1 (b), and the capacity is $\log(2 \cos(\frac{\pi}{D+1}))$ [9].

We now formally define this type of BRM code.

Definition 3 (BRM CODE WITH ONE OVERLAP AND CONSECUTIVE LEVELS $\mathcal{C}_I(n, m, D, 1)$) For the BRM code $\mathcal{C}_I(n, m, D, 1)$, every codeword $(P_1, P_2, \dots, P_{(n-1)/(m-1)})$ needs to satisfy the following additional constraint: the codeword has a realization $(B_1, B_2, \dots, B_{(n-1)/(m-1)})$ such that for $i = 1, 2, \dots, \frac{n-1}{m-1}$, the m cell levels in the block B_i form a set of m consecutive numbers. That is, if $B_i = (c'_1, c'_2, \dots, c'_m)$, then $\{c'_1, c'_2, \dots, c'_m\} = [\min_{j=1}^m c'_j, \max_{j=1}^m c'_j]$.

In a labeled graph for $\mathcal{C}_I(n, m, D, 1)$, each state corresponds to the charge level of an overlapped cell, so there are D states, $1, 2, \dots, D$. And each edge represents a permutation in a block (c'_1, \dots, c'_m) . The first (or last) digit in an edge labeling corresponds to the initial (or terminal) state of the edge. Let $(a_1, \dots, a_m) = \mathcal{I}(c'_1, \dots, c'_m)$, then $\forall k, l \in [1, m], c'_k - c'_l = a_k - a_l$. For example, the labeled graph for $\mathcal{C}_I(n, 3, 4, 1)$ is shown in Figure 1 (c).

The construction of the adjacency matrix for code $\mathcal{C}_I(n, m, D, 1)$ is presented in the following theorem.

Theorem 4 The adjacency matrix $A = (A_{ij})$ for $\mathcal{C}_I(n, m, D, 1)$ has

$$A_{ij} = \begin{cases} (m-2)! \min\{m - |i - j|, i, j, D - i + 1, \\ D - j + 1, D - m + 1\} & \text{if } 1 \leq |i - j| \leq m - 1, \text{ and } A_{ij} = 0 \text{ otherwise.} \end{cases} \quad (2)$$

if $1 \leq |i - j| \leq m - 1$, and $A_{ij} = 0$ otherwise.

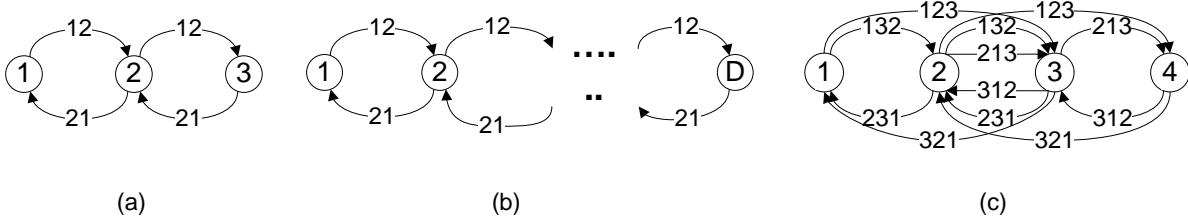


Fig. 1. Labeled graphs for \mathcal{C}_I . (a) $\mathcal{C}_I(n, 2, 3, 1)$; (b) $\mathcal{C}_I(n, 2, D, 1)$ and D is arbitrary; (c) $\mathcal{C}_I(n, 3, 4, 1)$.

Proof: A_{ij} indicates the number of permutations with $c'_1 = i, c'_m = j$. For fixed a_1 and a_m , there are $(m-2)!$ choices for (a_2, \dots, a_{m-1}) . Notice $i \rightarrow j$ only if $|a_1 - a_m| = |c'_1 - c'_m| \in [1, m-1]$. So $|\{(a_1, a_m)\}| \leq m - |i - j|$, if $|i - j| \in [1, m-1]$. And $|\{(a_1, a_m)\}| = 0$ otherwise. If $i \in [1, m]$, $\min_{1 \leq k \leq m} c'_k = c'_1 - (a_1 - 1) = i - a_1 + 1 \geq 1$, which implies $a_1 \in [1, i]$, or $|\{a_1\}| = i$. And we have similar results for other values of i . Hence, $|\{a_1\}| = \min\{i, D - i + 1, D - m + 1, m\}$. This argument also works for the terminal state j . Therefore, if $1 \leq |i - j| \leq m - 1$, then $A_{ij} = (m-2)! |\{(a_1, a_m)\}| = (m-2)! \min\{m - |i - j|, i, j, D - i + 1, D - j + 1, D - m + 1\}$. ■

The capacity of \mathcal{C}_I is $\text{cap}(\mathcal{C}_I) = \frac{\log \lambda(A)}{m-1}$. Some values of $\text{cap}(\mathcal{C}_I)$ and the capacity of the non-overlap code $\mathcal{C}(n, m, D, 0)$ (for comparison) are shown in Figure 2.

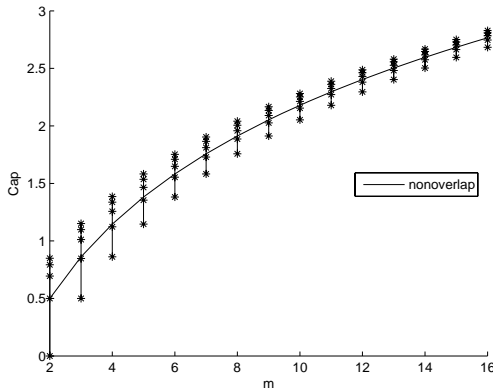


Fig. 2. Capacity for \mathcal{C}_I (stars) and for the non-overlap code (solid line). The stars in each vertical line correspond to the same permutation size m , and $D = m, m+1, \dots, m+4$ from bottom to top.

It is clear that the capacity of $\mathcal{C}_I(n, m, D, 1)$ increases with D . And if $D \rightarrow \infty$, $\text{cap}(\mathcal{C}_I(n, m, D, 1)) \rightarrow \frac{\log m!}{m-1}$, which is larger than the capacity of the non-overlap code. We now present a more general result.

Theorem 5 For any $m \geq 2$ and $D \geq m + 2$,

$$\text{cap}(\mathcal{C}_I(n, m, D, 1)) > \text{cap}(\mathcal{C}(n, m, D, 0))$$

Proof: Notice $\text{cap}(\mathcal{C}(n, m, D, 0)) = \log m! / m, \forall D \geq m$, so we need to prove $\text{cap}(\mathcal{C}_I(n, m, m+2, 1)) > \log m! / m$. When $m = 2, 3$, the theorem is trivial. When $m \geq 4, D = m + 2$, by (2), A is

$m \geq 4, D = m + 2$, by (2), A is

$$(m-2)! \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 0 & 0 \\ 1 & 0 & 2 & 2 & \dots & 2 & 1 & 0 \\ 1 & 2 & 0 & 3 & \dots & 3 & 2 & 1 \\ 1 & 2 & 3 & 0 & \dots & 3 & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & 3 & \dots & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 & \dots & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & \dots & 1 & 1 & 0 \end{pmatrix}_{(m+2) \times (m+2)}$$

Let $B = \frac{1}{(m-2)!} A$, I be the identity matrix and x be an indeterminate variable. $\det(B - xI) = 0$ implies $(-x - 3)^{m-3}(x^2 + x - 1)f(x) = 0$, where $f(x) = -x^3 + (3m - 8)x^2 + (7m - 10)x + 3m - 3$. Thus $\lambda(B)$ is the largest positive root of $f(x)$. It can be proven that $\lambda(B) > 3m - 6$, and $\lambda(A) > (3m - 6)(m - 2)!$. Now we are left to show $\frac{\log \lambda(A)}{m-1} > \frac{\log(3(m-2)(m-2)!)}{m-1} \geq \frac{\log m!}{m}$, which is equivalent to $\frac{3^m(m-2)^{m-1}(m-2)!}{m^{m-1}(m-1)^{m-1}} \geq 1$.

Notice $(1 - \frac{1}{m})^m \geq \frac{1}{e}$, and Stirling's Approximation, thus $\frac{3^m(m-2)^m(m-2)!}{m^{m-1}(m-1)^{m-1}} \geq \frac{1}{2e} \left(\frac{3}{e}\right)^m \sqrt{2\pi(m-1)} \geq 1$. ■

IV. BRM CODE WITH ONE OVERLAP

We now consider the general BRM code with one overlap, $\mathcal{C}(n, m, D, 1)$, which does not have the additional constraint of code $\mathcal{C}_I(n, m, D, 1)$.

The cell levels of a block, $\{c'_1, \dots, c'_m\}$, can be any set Q such that $Q \subseteq \{1, 2, \dots, D\}$ and $|Q| = m$. The labeled graph H generated is not deterministic in general. However, we are able to find a deterministic graph G that is equivalent to H [9]. Here is an example.

Example 6 The labeled graph H of $\mathcal{C}(n, 2, 4, 1)$ is shown in Figure 3 (a). This is not deterministic since state 1 has 3 outgoing edges labeled 12. Let G be the deterministic representation of \mathcal{C} , then the states $V(G)$ are subsets of $V(H)$. And for $u, v \in V(G)$, $u \xrightarrow{1} v$ if $\forall j \in v, \exists i \in u$ and $i \xrightarrow{1} j$. The resulting graph G is shown in Figure 3 (b). States $\{2\}, \{3\}, \{1, 3\}$, etc., have only outgoing edges, so their capacities are 0. Therefore the irreducible component of G maximizing $\lambda(A_i)$ is as in Figure 3 (c). Hence by (1) $\text{cap}(\mathcal{C}(n, 2, 4, 1)) = \log \lambda(A_i) = 0.8791$.

In general, suppose the deterministic graph G represents $\mathcal{C}(n, 2, D, 1)$, and A_i is the adjacency matrix

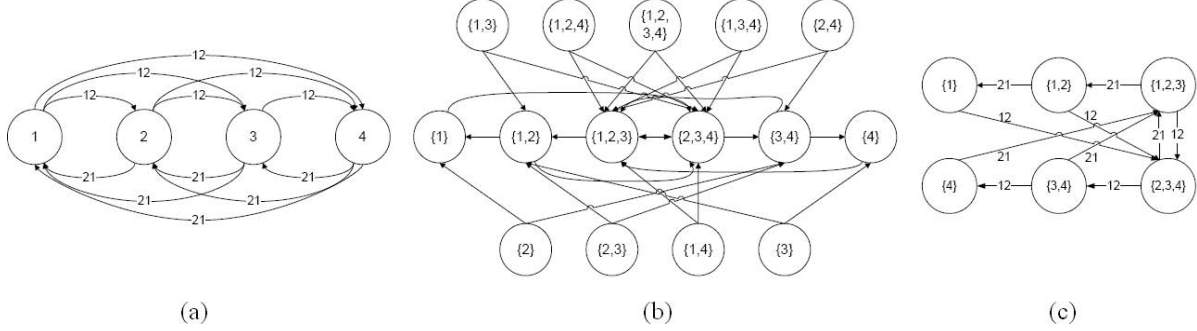


Fig. 3. Labeled graphs for $\mathcal{C}(n, 2, 4, 1)$. (a) Labeled graph; (b) deterministic graph; (c) irreducible graph.

for the irreducible component of G that has the largest eigenvalue. Then $\lambda(A_i)$ is the largest positive root of $-x^D + 2x^{D-1} - 1 = 0$. It can be seen that $\text{cap}(\mathcal{C})$ tends to 1 faster than $\text{cap}(\mathcal{C}_I)$ from the following table:

D	3	5	7	9	11
$\text{cap}(\mathcal{C}_I(n, 2, D, 1))$.5000	.7925	.8858	.9276	.9500
$\text{cap}(\mathcal{C}(n, 2, D, 1))$.6942	.9468	.9881	.9971	.9993

The construction in the above example can be naturally extended to the case $m > 2$.

Encoder/decoder for BRM codes can be constructed using sliding-block finite-state permutation encoder/decoder, cell-level encoder/decoder, and flash programming/reading. And the encoding rate can be arbitrarily close to the capacity. For example, a rate 3 : 4 block-decodable encoder can be constructed for $\mathcal{C}(n, 2, 4, 1)$. More details are shown in [11].

V. LOWER BOUND FOR CAPACITY

In this section, we present a lower bound to the capacity of the BRM code. To derive this, we first present a new form of rank modulation called the *star BRM*.

A. Star BRM

A Star BRM code uses $n + v$ cells. For convenience, let n be a multiple of $m - v$. v of these $n + v$ cells are called *anchors*, and we denote their cell levels by $(\ell_1, \ell_2, \dots, \ell_v)$. The other n cells are called *storage cells*, and we denote their cell levels by c_1, c_2, \dots, c_n . For $i = 1, 2, \dots, v$, $\ell_i \in [1, D]$; for $i = 1, 2, \dots, n$, $c_i \in [1, D]$. We call $(\ell_1, \ell_2, \dots, \ell_v, c_1, c_2, \dots, c_n)$ a *cell-level sequence*. For $i = 1, 2, \dots, \frac{n}{m-v}$, define block B_i as $(\ell_1, \ell_2, \dots, \ell_v, c_{(i-1)(m-v)+1}, c_{(i-1)(m-v)+2}, \dots, c_{i(m-v)})$. These $\frac{n}{m-v}$ blocks share the anchor cells. For $i = 1, 2, \dots, \frac{n}{m-v}$, we require that the m cell levels in B_i are all different, and let $P_i = \mathcal{I}(B_i)$. B_i is a *realization* of P_i . Again, a permutation sequence $(P_1, P_2, \dots, P_{n/(m-v)})$ may have multiple realizations.

Definition 7 (STAR BRM CODE $\mathcal{S}(n, m, D, v)$) In a Star BRM code $\mathcal{S}(n, m, D, v)$, every codeword is a permutation sequence $(P_1, P_2, \dots, P_{n/(m-v)})$ that has at₁₂₃₇

least one realization. Let $|\mathcal{S}(n, m, D, v)|$ denote the number of codewords in code \mathcal{S} . Then, the capacity is

$$\text{cap}(\mathcal{S}) = \lim_{n \rightarrow \infty} \frac{\log |\mathcal{S}(n, m, D, v)|}{n + v}.$$

To derive the capacity of Star BRM, we first show how the anchors $(\ell_1, \ell_2, \dots, \ell_v)$ affect the permutation sequences. For fixed $(\ell_1, \ell_2, \dots, \ell_v)$, define $Z(\ell_1, \ell_2, \dots, \ell_v)$ as the total number of permutations that can be induced by the cell levels $(\ell_1, \ell_2, \dots, \ell_v, c'_1, c'_2, \dots, c'_{m-v})$, where the m cell levels are all different and all belong to the set $[1, D]$. When we permute the v anchor levels, the value of $Z(\ell_1, \ell_2, \dots, \ell_v)$ remains the same. For example, when $v = 3$ and $D = 6$, $Z(2, 3, 6) = Z(3, 2, 6) = Z(6, 2, 3)$. So without loss of generality, assume $\ell_1 < \ell_2 < \dots < \ell_v$. Let $\beta(\ell_1, \ell_2, \dots, \ell_v)$ denote the number of solutions for the variables x_1, x_2, \dots, x_{v+1} such that (1) $\sum_{i=1}^{v+1} x_i = m - v$; (2) $x_1 \in [0, \ell_1 - 1]$, $x_i \in [0, \ell_i - \ell_{i-1} - 1]$ for $i \in [2, v]$, and $x_{v+1} \in [0, D - \ell_v]$.

Lemma 8. Given $D \geq m > v$, we have $Z(\ell_1, \ell_2, \dots, \ell_v) = (m - v)! \cdot \beta(\ell_1, \ell_2, \dots, \ell_v)$.

Sketch of the proof: A permutation induced by $(\ell_1, \ell_2, \dots, \ell_v, c'_1, c'_2, \dots, c'_{m-v})$ can be uniquely determined by the relative order of the $m - v$ cell levels $(c'_1, c'_2, \dots, c'_{m-v})$ and their relative values compared to $\ell_1, \ell_2, \dots, \ell_v$. ■

Lemma 9. $Z(\ell_1, \ell_2, \dots, \ell_v)$ is maximized when the numbers in the following set differ by at most one: $\{\ell_1 - 1, D - \ell_v\} \cup \{\ell_i - \ell_{i-1} - 1 | i = 2, 3, \dots, v\}$. (Every number in the above set is either $\lfloor \frac{D-v}{v+1} \rfloor$ or $\lceil \frac{D-v}{v+1} \rceil$.)

Please see [11] for detailed proofs of Lemma 8 and 9. Let $\ell_1^* < \ell_2^* < \dots < \ell_v^*$ be the v anchor levels that satisfy the condition in Lemma 9. and $Z^* = Z(\ell_1^*, \ell_2^*, \dots, \ell_v^*)$. Z^* can be computed using an algorithm of time complexity $O(D^2)$ (see [11]). The following theorem presents the capacity of the Star BRM.

Theorem 10. The capacity of $\mathcal{S}(n, m, D, v)$ is

$$\text{cap}(\mathcal{S}) = \frac{\log Z^*}{m - v}.$$

Sketch of the proof: $|\mathcal{S}(n, m, D, v)|$ is no less than the number of codewords induced by $(\ell_1^*, \ell_2^*, \dots, \ell_v^*, c_1, c_2, \dots, c_n)$, or $(Z^*)^{\frac{n}{m-v}}$. On the other hand, by Lemma 9, $|\mathcal{S}(n, m, D, v)|$ is no more than $(Z^*)^{\frac{n}{m-v}}$ times the number of choices for $(\ell_1, \ell_2, \dots, \ell_v)$, or $v! \binom{D}{v}$. Therefore, $\frac{\log Z^*}{m-v} \leq \text{cap}(\mathcal{S}) = \lim_{n \rightarrow \infty} \frac{\log |\mathcal{S}(n, m, D, v)|}{n+v} \leq \lim_{n \rightarrow \infty} \frac{\log(v! \binom{D}{v}) (Z^*)^{\frac{n}{m-v}}}{n} = \frac{\log Z^*}{m-v}$. So the theorem is proved. ■

The above proof leads to the following corollary.

Corollary 11 *The Star BRM code $\mathcal{S}(n, m, D, v)$ achieves its capacity even if the v anchor cell levels are fixed as $(\ell_1^*, \ell_2^*, \dots, \ell_v^*)$.*

The capacity of the Star BRM code $\mathcal{S}(n, m, D, v)$ is non-decreasing in D . However, when $D = (m - v + 1)v + (m - v)$, the capacity reaches its maximum value. Further increasing D will not increase the capacity. That is because when $D \geq (m - v + 1)v + (m - v)$, Z^* reaches its maximum value $m!/v!$.

B. Lower Bound for The Capacity of BRM

We now derive a lower bound for the capacity of the bounded rank modulation code $\mathcal{C}(n, m, D, v)$.

Theorem 12. *For the BRM code $\mathcal{C}(n, m, D, v)$, when $m \geq 2v$, its capacity*

$$\text{cap}(\mathcal{C}) \geq \frac{\log Z^* + \log v! + \log(m - 2v)!}{2(m - v)}.$$

Proof: Let $\mathcal{S}(n, m, D, v)$ be a Star BRM code such that every codeword has a realization in which the v anchors are $(\ell_1^*, \ell_2^*, \dots, \ell_v^*)$. By Corollary 11, \mathcal{S} achieves capacity.

For a codeword $s \in \mathcal{S}$, let $(\ell_1^*, \ell_2^*, \dots, \ell_v^*, c_1, c_2, \dots, c_n) = (B_1, B_2, \dots, B_{n/(m-v)})$ be its realization. For $i = 1, 2, \dots, n/(m - v)$, corresponding to block B_i , we build two blocks B_i' and B_i'' of length m as follows. Say $B_i = (\ell_1^*, \ell_2^*, \dots, \ell_v^*, c_1', c_2', \dots, c_{m-v}')$. The first v cell levels of B_i' take values from the set $\{\ell_1^*, \ell_2^*, \dots, \ell_v^*\}$ (we have $v!$ choices), and the next $m - v$ cell levels of B_i' are the same as $(c_1', c_2', \dots, c_{m-v}')$. The first v cell levels of B_i'' overlap the last v cell levels of B_i' . We pick $m - 2v \leq D - 2v$ values different from the first v and the last v cell levels of B_i' , and assign them to the next $m - 2v$ cell levels of B_i'' (we have $(m - 2v)!$ choices). The final v cell levels of B_i'' take values again from the set $\{\ell_1^*, \ell_2^*, \dots, \ell_v^*\}$. Then we construct a cell-level sequence $(B_1', B_1'', B_2', B_2'', \dots, B_{n/(m-v)}', B_{n/(m-v)}'')$, where every two adjacent blocks overlap by v . Corresponding to every codeword $s \in \mathcal{S}$, there are at least $(v!(m - 2v)!)^{\frac{n}{m-v}}$ such cell-level sequences, which we denote by Q_s . No two cell-level sequences in Q_s induce the same permutation sequence. And when $s \neq s'$, every pair of cell-level sequences

from Q_s and $Q_{s'}$, respectively, also induce different permutation sequences. Besides, every cell-level sequence constructed above induces a codeword in the code $\mathcal{C}(2n + v, m, D, v)$.

So corresponding to the $|\mathcal{S}(n, m, D, v)|$ codewords of the Star BRM code $\mathcal{S}(n, m, D, v)$, we can find at least $|\mathcal{S}(n, m, D, v)|(v!(m - 2v)!)^{\frac{n}{m-v}}$ codewords of the BRM code $\mathcal{C}(2n + v, m, D, v)$. So the capacity of code $\mathcal{C}(n, m, D, v)$ is $\text{cap}(\mathcal{C}) \geq \lim_{n \rightarrow \infty} \frac{\log |\mathcal{S}(n, m, D, v)| + (\log v! + \log(m - 2v)!) \cdot \frac{n}{m-v}}{2n+v} = \frac{\log Z^* + \log v! + \log(m - 2v)!}{2(m-v)}$. So the theorem is proved. ■

Corollary 13 *Let $\mathcal{C}(n, m, D, v)$ be a BRM code, and let $\mathcal{S}(n, m, D, v)$ be a Star BRM code. Then, when $m \geq 2v$,*

$$\text{cap}(\mathcal{C}) \geq \frac{1}{2} \cdot \text{cap}(\mathcal{S}).$$

In particular, if $v > 1$ or $m > 2v$, $\text{cap}(\mathcal{C}) > \frac{1}{2} \cdot \text{cap}(\mathcal{S})$.

Define $A_k^n = \binom{n}{k} k! = n! / (n - k)!$. Suppose $m < 2v$ and $v = k(m - v) + s$, where $k \in \mathbb{N}^+$ and $1 \leq s \leq m - v$. Let $r = m - v - s$. Define a constant $M = A_s^{m-v} (A_{m-v}^{2(m-v)-s})^{k-1} (m - v)!$. Similar to Theorem 12, we have the following lower bound for the BRM code when $m < 2v$ (see [11] for proof).

Theorem 14 *For the BRM code $\mathcal{C}(n, m, D, v)$, when $m < 2v$ and $D \geq m + r$, its capacity*

$$\text{cap}(\mathcal{C}) \geq \frac{\log(Z^* \cdot M \cdot r!)}{m + r}$$

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