Sparse Matrix Methods
Chapter 5 lecture notes
QR factorization

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Householder reflections

- $H = I - \beta vv^T$; β a scalar and v a vector
- can choose v and β based on x so that Hx is all zero except $(Hx)_1 = \pm\|x\|_2$.
- $Hx = x - v(\beta(v^Tx))$
- H is symmetric ($H = H^T$)
- H is orthogonal $HH^T = H^TH = I$
- if x_1 is nonzero, then x and v have the *same* nonzero pattern ($\mathcal{V} = \mathcal{X}$).
- if x_1 is zero, then permute the rows of x to make it so

QR factorization

- pick `H1` based on `A(:,1)`, to zero out all but `A(1,1)`
- then `A = H1 * A`
- repeat for 2nd column, zeroing out everything below the diagonal
- etc, until `A` becomes upper triangular

Theorem (Golub)

The QR factorization $QR = A$, where $A \in \mathbb{R}^{m \times n}$ and $m \geq n$, is
$Q = H_1 H_2 \cdots H_n = \prod_{k=1}^{n} H_k$ *and*
$R = Q^T A = H_n \cdots H_2 H_1 A = (\prod_{k=n}^{1} H_k) A = A^{[n]}$.

MATLAB prototype: right-looking QR

```
function [V,Beta,R] = qr_right (A)
[m n] = size (A) ;
V = zeros (m,n) ;
Beta = zeros (1,n) ;
for k = 1:n
    [v,beta,s] = gallery ('house', A (k:m,k), 2) ;
    V (k:m,k) = v ;
    Beta (k) = beta ;
    A (k:m,k:n) = A (k:m,k:n) - v * (beta * (v' * A (k:m,k:n))) ;
end
R = A ;
```

MATLAB prototype: left-looking QR

```
function [V,Beta,R] = qr_left (A)
[m n] = size (A) ;
V = zeros (m,n) ;
Beta = zeros (1,n) ;
R = zeros (m,n) ;
for k = 1:n
    x = A (:,k) ;
    for i = 1:k-1
        v = V (i:m,i) ;
        beta = Beta (i) ;
        x (i:m) = x (i:m) - v * (beta * (v' * x (i:m))) ;
    end
    [v,beta,s] = gallery ('house', x (k:m), 2) ;
    V (k:m,k) = v ;
    Beta (k) = beta ;
    R (1:(k-1),k) = x (1:(k-1)) ;
    R (k,k) = s ;
end
```

Nonzero pattern of HA

Theorem (George, Liu, and Ng)

Consider $HA = A - v(\beta(v^T A))$. Then $(HA)_{i*}$ where $i \notin \mathcal{V}$ is equal to row i of A. For any row $i \in \mathcal{V}$, the nonzero pattern of $(HA)_{i*}$ is

$$\bigcup_{i \in \mathcal{V}} \mathcal{A}_{i*}. \tag{1}$$

That is, in HA, the nonzero pattern of any modified row $i \in \mathcal{V}$ is replaced with the set union of all rows that are modified by the Householder reflection H.

Nonzero pattern of R

Theorem (Golub and Van Loan [**?**])

If A^TA is positive definite, and its Cholesky factorization is $LL^T = A^TA$, then $L = R^T_{1\ldots n,1\ldots n}$.

Nonzero pattern of R, continued

Theorem (Coleman et al., George and Heath)

Assuming the matrix A has the strong Hall property, $\mathcal{R}_{*k} = \mathcal{L}_{k*}$, where $\mathcal{L}_{k*}$ denotes the nonzero pattern of the kth row of the symbolic Cholesky factor of $A^T A$. If A does not have the strong Hall property, $\mathcal{R}_{*k} \subseteq \mathcal{L}_{k*}$.

More concisely ...

Theorem

$\mathcal{R}_{*k} = Reach_{\mathcal{T}_k}(\{\min \mathcal{A}_{i*} \mid i \in \mathcal{A}_{*k}\})$ *(assuming A has the strong Hall property).*

Nonzero pattern of the Householder vectors

Theorem

$$\mathcal{V}_k = \left(\bigcup_{k=parent(i)} \mathcal{V}_i \setminus \{i\} \right) \cup \{i \mid k = \min \mathcal{A}_{i*}\},$$

where each set in the above expression is disjoint from all other sets, and A has the strong Hall property. That is,

$$|\mathcal{V}_k| = \left(\sum_{k=parent(i)} |\mathcal{V}_i| - 1 \right) + |\{i \mid k = \min \mathcal{A}_{i*}\}| \qquad (2)$$

If A does not have the strong Hall property, this is an upper bound on $\mathcal{V}$.

left-looking QR

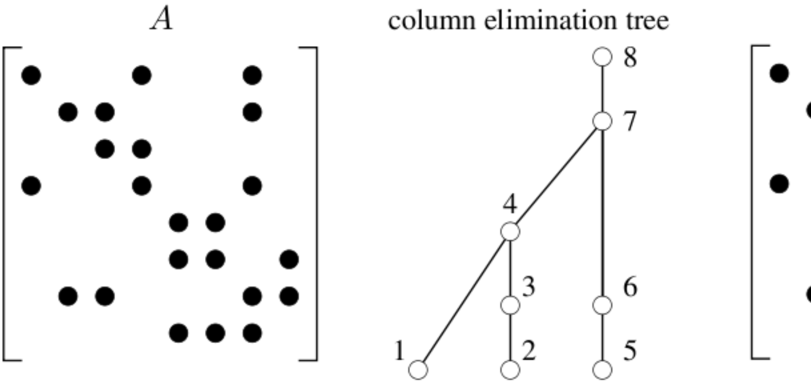

Figure 5.1. *QR factorization*

left-looking QR

function $[V, \beta_{1\ldots n}, R] = \texttt{sparse\_qr\_left}(A)$
    $\mathcal{T}$ = elimination tree of $A^T A$
    compute $|R|$ using `cs_counts` of $A^T A$
    compute $|\mathcal{V}_{1\ldots n}|$ using (5.2)
    for $k = 0$ to $n-1$ **do**
        $\mathcal{R}_{*k} = \mathrm{Reach}_{\mathcal{T}_k}(\{\min \mathcal{A}_{i*} \mid i \in \mathcal{A}_{*k}\})$
        $x = A_{*k}$
        $\mathcal{V}_k = \mathcal{A}_{*k}$
        for each $i \in \mathcal{R}_{*k}$ **do**
            $x = x - v_i(\beta_i(v_i^T x))$
            if $\texttt{parent}(i) = k$ **then**
                $\mathcal{V}_k = \mathcal{V}_k \cup \mathcal{V}_i \setminus \{i\}$
        $R_{1\ldots k-1,k} = x_{1\ldots k-1}$
        $[v_k, \beta_k, r_{kk}] = \texttt{house}(x_{k\ldots m})$