Proof by Induction

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Based on slides by Andreas Klappenecker
Motivation

Induction is an axiom which allows us to prove that certain properties are true for all positive integers (or for all nonnegative integers, or all integers $\geq$ some fixed number)
Induction Principle

Let $A(n)$ be an assertion concerning the integer $n$.

If we want to show that $A(n)$ holds for all positive integer $n$, we can proceed as follows:

**Induction basis:** Show that the assertion $A(1)$ holds.

**Induction step:** For all positive integers $n$, show that $A(n)$ implies $A(n+1)$. 
Standard Example

For all positive integers \( n \), we have

\[
A(\text{n}): 1+2+...+n = \frac{n(n+1)}{2}
\]

**Induction basis:**

Since \( 1 = \frac{1(1+1)}{2} \), the assertion \( A(1) \) is true.

**Induction step:**

Suppose that \( A(n) \) holds. Then

\[
1+2+...+n+(n+1) = \frac{n(n+1)}{2} + n+1 = \frac{n^2 + n+2n+2}{2} = \frac{(n+1)(n+2)}{2},
\]

hence \( A(n+1) \) holds. Therefore, the claim follows by induction on \( n \).
The Main Points

We established in the induction basis that the assertion $A(1)$ is true.

We showed in the induction step that $A(n+1)$ holds, assuming that $A(n)$ holds.

In other words, we showed in the induction step that $A(n) \rightarrow A(n+1)$ holds for all $n \geq 1$. 
Theorem: For all positive integers $n$, we have

$$1+3+5+...+(2n-1) = n^2$$

Proof. We prove this by induction on $n$. Let $A(n)$ be the assertion of the theorem.

Induction basis: Since $1 = 1^2$, it follows that $A(1)$ holds.

Induction step: Suppose that $A(n)$ holds. Then

$$1+3+5+...+(2n-1)+(2n+1) = n^2+2n+1 = (n+1)^2$$

holds. In other words, $A(n)$ implies $A(n+1)$. 
Theorem: We have
\[1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}\]
for all \(n \geq 1\).

Proof. Your turn!!!

Let \(B(n)\) denote the assertion of the theorem.

Induction basis:

Since \(1^2 = 1(1+1)(2+1)/6\), we can conclude that \(B(1)\) holds.
Inductive step: Suppose that $B(n)$ holds. Then

$$1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2$$

Factoring out $(n+1)$ on the right hand side yields

$$(n+1)(n(2n+1)+6(n+1))/6 = (n+1)(2n^2 +7n+6)/6$$

One easily verifies that this is equal to

$$(n+1)(((n+1)+1)(2(n+1)+1)/6$$

Thus, $B(n+1)$ holds.

Therefore, the proof follows by induction on $n$. 
Tip

How can you verify whether your algebra is correct?

Use http://www.wolframalpha.com

[Not allowed in any exams, though. Sorry!]
What's Wrong?
Billiard Balls

“**Theorem**”: All billiard balls have the same color.

Proof: By induction, on the number of billiard balls.

**Induction basis:**

Our theorem is certainly true for $n=1$.

**Induction step:**

Assume the theorem holds for $n$ billiard balls. We prove it for $n+1$. Look at the first $n$ billiard balls among the $n+1$. By induction hypothesis, they have the same color. Now look at the last $n$ billiard balls. They have the same color. Hence all $n+1$ billiard balls have the same color.
"Theorem": For all positive integers \( n \), we have \( n = n + 1 \).

"Proof": Suppose that the claim is true for \( n = k \). Then
\[
k + 1 = (k) + 1 = (k + 1) + 1
\]
by induction hypothesis. Thus, \( k + 1 = k + 2 \).

Therefore, the theorem follows by induction on \( n \).
Maximally Weird!

"Theorem": For all positive integers n, if a and b are positive integers such that \( \max\{a,b\}=n \), then \( a=b \).

Proof: By induction on n. The result holds for \( n = 1 \), i.e., if \( \max\{a, b\} = 1 \), then \( a = b = 1 \).

Suppose it holds for \( n \), i.e., if \( \max\{a, b\} = n \), then \( a = b \). Now consider \( \max\{a, b\} = n + 1 \).

**Case 1:** \( a - 1 \geq b - 1 \). Then \( a \geq b \). Hence \( a = \max\{a, b\} = n + 1 \).

Thus \( a - 1 = n \) and \( \max\{a - 1, b - 1\} = n \).

By induction hypothesis, \( a-1=b-1 \). Hence \( a=b \).

**Case 2:** \( b - 1 \geq a - 1 \).

Same argument.
Fallacy: In the proof we used the inductive hypothesis to conclude \( \max\{a - 1, b - 1\} = n \Rightarrow a - 1 = b - 1 \). However, we can only use the inductive hypothesis if \( a - 1 \) and \( b - 1 \) are positive integers. This does not have to be the case as the example \( b=1 \) shows.
More Examples
Factorials

Theorem. \[ \sum_{i=0}^{n} i(i!) = (n + 1)! - 1. \]

By convention: 0! = 1

Induction basis:
Since 0 = 1 − 1, the claim holds for \( n = 0 \).

Induction Step:
Suppose the claim is true for \( n \). Then
\[
\sum_{i=0}^{n+1} i(i!) = (n + 1)(n + 1)! + \sum_{i=0}^{n} i(i!)
\]
\[
= (n + 1)(n + 1)! + (n + 1)! - 1 \text{ by ind. hyp.}
\]
\[
= (n + 2)(n + 1)! - 1
\]
\[
= (n + 2)! - 1
\]
Theorem: For all positive integers $n$, the number $7^n - 2^n$ is divisible by 5.

Proof: By induction.

Induction basis. Since $7 - 2 = 5$, the theorem holds for $n = 1$. 

Divisibility
Divisibility

Inductive step:

Suppose that $7^n - 2^n$ is divisible by 5. Our goal is to show that this implies that $7^{n+1} - 2^{n+1}$ is divisible by 5. We note that

$$7^{n+1} - 2^{n+1} = 7^n - 2^n + 2^n = 5x7^n + 2x7^n - 2x2^n = 5x7^n + 2(7^n - 2^n).$$

By induction hypothesis, $(7^n - 2^n) = 5k$ for some integer $k$.

Hence, $7^{n+1} - 2^{n+1} = 5x7^n + 2x5k = 5(7^n + 2k)$, so

$7^{n+1} - 2^{n+1} = 5 \times$ some integer.

Thus, the claim follows by induction on $n$. 
Strong Induction
Suppose we wish to prove a certain assertion concerning positive integers.

Let $A(n)$ be the assertion concerning the integer $n$.

To prove it for all $n \geq 1$, we can do the following:

1) Prove that the assertion $A(1)$ is true.

2) Assuming that the assertions $A(k)$ are proved for all $k < n$, prove that the assertion $A(n)$ is true.

We can conclude that $A(n)$ is true for all $n \geq 1$. 

Strong Induction
Strong Induction

Induction basis:
Show that $A(1)$ is true.

Induction step:
Show that $(A(1) \land A(2) \land \ldots \land A(n)) \rightarrow A(n+1)$ holds for all $n \geq 1$.

---

21

strong induction hypothesis
Postage

**Theorem:** Every amount of postage that is at least 12 cents can be made from 4¢ and 5¢ stamps.
Postage

Proof by induction on the amount of postage.

**Induction Basis:** If the postage is

- 12¢: use three 4¢ and zero 5¢ stamps ($12 = 3 \times 4 + 0 \times 5$)
- 13¢: use two 4¢ and one 5¢ stamps ($13 = 2 \times 4 + 1 \times 5$)
- 14¢: use one 4¢ and two 5¢ stamps ($14 = 1 \times 4 + 2 \times 5$)
- 15¢: use zero 4¢ and three 5¢ stamps ($15 = 0 \times 4 + 3 \times 5$)

(Not part of induction basis, but let us try some more)

- 16¢: use (three+one) 4¢ and zero 5¢ stamps (($3+1) \times 4 + 0 \times 5$)
- 17¢: use (two+one) 4¢ and one 5¢ stamps (($2+1) \times 4 + 1 \times 5$)
- 18¢: use (one+one) 4¢ and two 5¢ stamps (($1+1) \times 4 + 2 \times 5$)
- 19¢: use (zero+one) 4¢ and three 5¢ stamps (($0+1) \times 4 + 3 \times 5$)
- 20¢: use (three+two) 4¢ and zero 5¢ stamps (($3+2) \times 4 + 0 \times 5$)

...
Postage

Inductive step:
Suppose that we have shown how to construct postage for every value from 12 up through $k$. We need to show how to construct $k + 1$ cents of postage.

Since we’ve already proved the induction basis, we may assume that $k + 1 \geq 16$. Since $k+1 \geq 16$, we have $(k+1)-4 \geq 12$. By inductive hypothesis, we can construct postage for $(k + 1) - 4$ cents using $m$ 4¢ stamps and $n$ 5¢ stamps for some non-negative integers $m$ and $n$. In other words $((k + 1) - 4) = 4m + 5n$; hence, $k+1 = 4(m+1)+5n$. 

24
Why did we need to establish four cases in the induction basis?

Isn’t it enough to remark that the postage for 12 cents is given by three 4 cents stamps?
Another Example: Sequence

Theorem: Let a sequence \((a_n)\) be defined as follows:

\[
\begin{align*}
a_0 &= 1, \\
a_1 &= 2, \\
a_2 &= 3, \\
a_k &= a_{k-1} + a_{k-2} + a_{k-3} \quad \text{for all integers } k \geq 3.
\end{align*}
\]

Then \(a_n \leq 2^n\) for all integers \(n \geq 0\). \(P(n)\)

Proof. Induction basis:

The statement is true for \(n=0\), since \(a_0 = 1 \leq 1 = 2^0\) \(P(0)\)

for \(n=1\): since \(a_1 = 2 \leq 2 = 2^1\) \(P(1)\)

for \(n=2\): since \(a_2 = 3 \leq 4 = 2^2\) \(P(2)\)
Sequence (cont’d)

Inductive step:

(S.I.H.) Assume that \( P(i) \) is true for all \( i \) with \( 0 \leq i < k \), that is, \( a_i \leq 2^i \) for all \( 0 \leq i < k \), where \( k > 2 \).

Show that \( P(k) \) is true: \( a_k \leq 2^k \)

\[
\begin{align*}
a_k &= a_{k-1} + a_{k-2} + a_{k-3} \\
&\leq 2^{k-1} + 2^{k-2} + 2^{k-3} \\
&\leq 2^0 + 2^1 + \ldots + 2^{k-3} + 2^{k-2} + 2^{k-1} \\
&= 2^{k-1} \leq 2^k \quad \text{by understanding binary number system}
\end{align*}
\]

Thus, \( P(n) \) is shown true for all integers \( n \geq 0 \) by strong induction.
Yet Another Example Sequence

A sequence $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

\[
\begin{align*}
a_0 &= 0; \\
a_1 &= 1; \\
a_n &= 5a_{n-1} - 6a_{n-2} \text{ for all } n \geq 2.
\end{align*}
\]

Prove that for all non-negative integers $n$, $a_n = 3^n - 2^n : P(n)$

Proof. \textbf{Induction basis}: need - to show $P(0)$ and $P(1)$ hold.

$P(0)$ holds since $a_0 = 0 = 3^0 - 2^0$

$P(1)$ holds since $a_1 = 1 = 3^1 - 2^1$
Yet Another Example Sequence (Cont.)

Inductive step:

(S.I.H.) Assume that $P(i)$ is true for all $i$ with $0 \leq i < n$, that is, $a_i = 3^i - 2^i$ for all $0 \leq i < n$, where $n > 1$.

Show that $P(n)$ is true:

$$a_n = 3^n - 2^n$$

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{by def. of seq.}$$

$$= 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2}) \quad \text{by S.I.H.}$$

$$= (3+2)(3^{n-1} - 2^{n-1}) - 3 \cdot 2(3^{n-2} - 2^{n-2})$$

$$= 3 \cdot 3^{n-1} - 3 \cdot 2^{n-1} + 2 \cdot 3^{n-1} - 2 \cdot 2^{n-1} - 2 \cdot 3^{n-2} + 3 \cdot 2 \cdot 2^{n-2}$$

$$= 3 \cdot 3^{n-1} - 2 \cdot 2^{n-1} = 3^n - 2^n$$

Thus, $P(n)$ is shown true for all integers $n \geq 0$ by strong induction.