Induction is an axiom which allows us to prove that certain properties are true for all positive integers (or for all nonnegative integers, or all integers $\geq$ some fixed number)
Induction Principle

Let $A(n)$ be an assertion concerning the integer $n$.

If we want to show that $A(n)$ holds for all positive integer $n$, we can proceed as follows:

**Induction basis:** Show that the assertion $A(1)$ holds.

**Induction step:** For all positive integers $n$, show that $A(n)$ implies $A(n+1)$. 
Standard Example

For all positive integers \( n \), we have

\[
A(n): \quad 1+2+\ldots+n = \frac{n(n+1)}{2}
\]

**Induction basis:**

Since \( 1 = 1(1+1)/2 \), the assertion \( A(1) \) is true.

**Induction step:**

Suppose that \( A(n) \) holds. Then

\[
1+2+\ldots+n+(n+1) = \frac{n(n+1)}{2} + n+1 = \frac{n^2 + n+2n+2}{2}
= \frac{(n+1)(n+2)}{2},
\]

hence \( A(n+1) \) holds. Therefore, the claim follows by induction on \( n \).
The Main Points

We established in the induction basis that the assertion $A(1)$ is true.

We showed in the induction step that $A(n+1)$ holds, assuming that $A(n)$ holds.

In other words, we showed in the induction step that $A(n) \rightarrow A(n+1)$ holds for all $n \geq 1$. 
**Example 2**

**Theorem:** For all positive integers \( n \), we have

\[ 1 + 3 + 5 + \ldots + (2n-1) = n^2 \]

**Proof.** We prove this by induction on \( n \). Let \( A(n) \) be the assertion of the theorem.

**Induction basis:** Since \( 1 = 1^2 \), it follows that \( A(1) \) holds.

**Induction step:** Suppose that \( A(n) \) holds. Then

\[ 1 + 3 + 5 + \ldots + (2n-1) + (2n+1) = n^2 + 2n + 1 = (n+1)^2 \]

holds. In other words, \( A(n) \) implies \( A(n+1) \).
Theorem: We have

\[1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}\]

for all \(n \geq 1\).

Proof. Your turn!!!

Let \(B(n)\) denote the assertion of the theorem.

Induction basis:

Since \(1^2 = \frac{1(1+1)(2+1)}{6}\), we can conclude that \(B(1)\) holds.
**Inductive step:** Suppose that $B(n)$ holds. Then
\[1^2 + 2^2 + ... + n^2 + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2\]
Factoring out $(n+1)$ on the right hand side yields
\[(n+1)(n(2n+1)+6(n+1))/6 = (n+1)(2n^2 +7n+6)/6\]
One easily verifies that this is equal to
\[(n+1)((n+1)+1)(2(n+1)+1)/6\]
Thus, $B(n+1)$ holds.

Therefore, the claim follows by induction on $n$. 
Tip

How can you verify whether your algebra is correct?

Use http://www.wolframalpha.com

[Not allowed in any exams, though. Sorry!]
What's Wrong?
“**Theorem**”: All billiard balls have the same color.

Proof: By induction, on the number of billiard balls.

Induction basis:

Our theorem is certainly true for \( n = 1 \).

Induction step:

Assume the theorem holds for \( n \) billiard balls. We prove it for \( n+1 \). Look at the first \( n \) billiard balls among the \( n+1 \). By induction hypothesis, they have the same color. Now look at the last \( n \) billiard balls. They have the same color. Hence all \( n+1 \) billiard balls have the same color.
Weird Properties of Positive Integers

"Theorem": For all positive integers $n$, we have $n=n+1$.

"Proof": Suppose that the claim is true for $n=k$. Then

$$k+1 = (k) + 1 = (k+1) + 1$$

by induction hypothesis. Thus, $k+1=k+2$.

Therefore, the theorem follows by induction on $n$.

What’s wrong?
Maximally Weird!

“**Theorem**“: For all positive integers n, if a and b are positive integers such that \( \max\{a,b\}=n \), then \( a=b \).

Proof: By induction on \( n \). The result holds for \( n = 1 \), i.e., if \( \max\{a, b\} = 1 \), then \( a = b = 1 \).

Suppose it holds for \( n \), i.e., if \( \max\{a,b\} = n \), then \( a = b \). Now consider \( \max\{a, b\} = n + 1 \).

**Case 1**: \( a - 1 \geq b - 1 \). Then \( a \geq b \). Hence \( a=\max\{a,b\}=n+1 \).

Thus \( a - 1 = n \) and \( \max\{a - 1, b - 1\} = n \).

By induction hypothesis, \( a-1=b-1 \). Hence \( a=b \).

**Case 2**: \( b - 1 \geq a - 1 \).

Same argument.
Fallacy: In the proof we used the inductive hypothesis to conclude $\max\{a - 1, b - 1\} = n \Rightarrow a - 1 = b - 1$. However, we can only use the inductive hypothesis if $a - 1$ and $b - 1$ are positive integers. This does not have to be the case as the example $b=1$ shows.
More Examples
Theorem. \[ \sum_{i=0}^{n} i(i!) = (n + 1)! - 1. \]

By convention: 0! = 1

Induction basis:
Since 0 = 1 - 1, the claim holds for \( n = 0 \).

Induction Step:
Suppose the claim is true for \( n \). Then
\[
\sum_{i=0}^{n+1} i(i!) = (n + 1)(n + 1)! + \sum_{i=0}^{n} i(i!)
\]
\[
= (n + 1)(n + 1)! + (n + 1)! - 1 \text{ by ind. hyp.}
\]
\[
= (n + 2)(n + 1)! - 1
\]
\[
= (n + 2)! - 1
\]
Divisibility

**Theorem**: For all positive integers $n$, the number $7^n - 2^n$ is divisible by 5.

**Proof**: By induction.

*Induction basis*. Since $7 - 2 = 5$, the theorem holds for $n=1$. 
Divisibility

Inductive step:

Suppose that $7^n - 2^n$ is divisible by 5. Our goal is to show that this implies that $7^{n+1} - 2^{n+1}$ is divisible by 5. We note that

$$7^{n+1} - 2^{n+1} = 7 \cdot 7^n - 2 \cdot 2^n = 5 \cdot 7^n + 2 \cdot 7^n - 2 \cdot 2^n = 5 \cdot 7^n + 2(7^n - 2^n).$$

By induction hypothesis, $(7^n - 2^n) = 5k$ for some integer $k$. Hence, $7^{n+1} - 2^{n+1} = 5 \cdot 7^n + 2 \cdot 5k = 5(7^n + 2k)$, so

$$7^{n+1} - 2^{n+1} = 5 \times \text{some integer.}$$

Thus, the claim follows by induction on $n$. 
Strong Induction
Strong Induction

Suppose we wish to prove a certain assertion concerning positive integers.

Let $A(n)$ be the assertion concerning the integer $n$.

To prove it for all $n \geq 1$, we can do the following:

1) Prove that the assertion $A(1)$ is true.

2) Assuming that the assertions $A(k)$ are proved for all $k < n$, prove that the assertion $A(n)$ is true.

We can conclude that $A(n)$ is true for all $n \geq 1$. 
Strong Induction

**Induction basis:**
Show that $A(1)$ is true.

**Induction step:**
Show that $(A(1) \land A(2) \land ... \land A(n)) \rightarrow A(n+1)$ holds for all $n \geq 1$. 

**strong induction hypothesis**
Theorem: Every amount of postage that is at least 12 cents can be made from 4¢ and 5¢ stamps.
Postage

Proof by induction on the amount of postage.

**Induction Basis:** If the postage is
12¢: use three 4¢ and zero 5¢ stamps (12=3×4+0×5)
13¢: use two 4¢ and one 5¢ stamps (13=2×4+1×5)
14¢: use one 4¢ and two 5¢ stamps (14=1×4+2×5)
15¢: use zero 4¢ and three 5¢ stamps (15=0×4+3×5)

(Not part of induction basis, but let us try some more)
16¢: use (three+one) 4¢ and zero 5¢ stamps ((3+1)×4+0×5)
17¢: use (two+one) 4¢ and one 5¢ stamps ((2+1)×4+1×5)
18¢: use (one+one) 4¢ and two 5¢ stamps ((1+1)×4+2×5)
19¢: use (zero+one) 4¢ and three 5¢ stamps ((0+1)×4+3×5)
20¢: use (three+two) 4¢ and zero 5¢ stamps ((3+2)×4+0×5)

...
Postage

Inductive step:
Suppose that we have shown how to construct postage for every value from 12 up through k. We need to show how to construct k + 1 cents of postage.

Since we’ve already proved the induction basis, we may assume that k + 1 ≥ 16. Since k + 1 ≥ 16, we have (k + 1) - 4 ≥ 12. By inductive hypothesis, we can construct postage for (k + 1) - 4 cents using m 4¢ stamps and n 5¢ stamps for some non-negative integers m and n. In other words, (k + 1) - 4 = 4m + 5n; hence, k + 1 = 4(m + 1) + 5n.
Why did we need to establish four cases in the induction basis?

Isn’t it enough to remark that the postage for 12 cents is given by three 4 cents stamps?
Another Example: Sequence

Theorem: Let a sequence \((a_n)\) be defined as follows:

\[ a_0 = 1, \quad a_1 = 2, \quad a_2 = 3, \]
\[ a_k = a_{k-1} + a_{k-2} + a_{k-3} \] for all integers \(k \geq 3\).

Then \(a_n \leq 2^n\) for all integers \(n \geq 0\). \(P(n)\)

Proof. Induction basis:

The statement is true for \(n=0\), since \(a_0 = 1 \leq 1 = 2^0 \) \(P(0)\)

for \(n=1\): since \(a_1 = 2 \leq 2 = 2^1 \) \(P(1)\)

for \(n=2\): since \(a_2 = 3 \leq 4 = 2^2 \) \(P(2)\)
Sequence (cont’d)

Inductive step:

(S.I.H.) Assume that $P(i)$ is true for all $i$ with $0 \leq i < k$, that is, $a_i \leq 2^i$ for all $0 \leq i < k$, where $k > 2$.

Show that $P(k)$ is true: $a_k \leq 2^k$

$a_k = a_{k-1} + a_{k-2} + a_{k-3}$ by def. of seq.

$\leq 2^{k-1} + 2^{k-2} + 2^{k-3}$ by S.I.H.

$\leq 2^0 + 2^1 + \cdots + 2^{k-3} + 2^{k-2} + 2^{k-1}$

$= 2^{k-1} \leq 2^k$ by understanding binary number system

Thus, $P(n)$ is shown true for all integers $n \geq 0$ by strong induction.
Yet Another Example Sequence

A sequence $a_0, a_1, a_2, \ldots$ is defined recursively as follows:

$$a_0 = 0;$$
$$a_1 = 1;$$
$$a_n = 5a_{n-1} - 6a_{n-2} \text{ for all } n \geq 2.$$ 

Prove that for all non-negative integers $n$, $a_n = 3^n - 2^n : P(n)$

Proof. Induction basis: need to show $P(0)$ and $P(1)$ hold.

$P(0)$ holds since $a_0 = 0 = 1 - 1 = 3^0 - 2^0$

$P(1)$ holds since $a_1 = 1 = 3 - 2 = 3^1 - 2^1$
Inductive step:

(S.I.H.) Assume that $P(i)$ is true for all $i$ with $0 \leq i < n$, that is, $a_i = 3^i - 2^i$ for all $0 \leq i < n$, where $n > 1$.

Show that $P(n)$ is true: $a_n = 3^n - 2^n$

$a_n = 5a_{n-1} - 6a_{n-2}$ by def. of seq.

$= 5(3^{n-1} - 2^{n-1}) - 6(3^{n-2} - 2^{n-2})$ by S.I.H.

$= (3+2)(3^{n-1} - 2^{n-1}) - 3\cdot2(3^{n-2} - 2^{n-2})$

$= 3\cdot3^{n-1} - 3\cdot2^{n-1} + 2\cdot3^{n-1} - 2\cdot2^{n-1} - 2\cdot3^{n-2} + 3\cdot2\cdot2^{n-2}$

$= 3\cdot3^{n-1} - 2\cdot2^{n-1} = 3^n - 2^n$

Thus, $P(n)$ is shown true for all integers $n \geq 0$ by strong induction.