In this short lecture note, I will explain the difference between proof by contrapositive and proof by contradiction, which seem to cause easily some confusions. The basic concept is that proof by contrapositive relies on the fact that \( p \rightarrow q \) and its contrapositive \( \neg q \rightarrow \neg p \) are logically equivalent, thus, if \( p(x) \rightarrow q(x) \) is true for all \( x \) then \( \neg q(x) \rightarrow \neg p(x) \) is also true for all \( x \), vice versa. On the other hand, proof by contradiction relies on the simple fact that if the given theorem \( P \) is true, the \( \neg P \) is not true. This proof method is applied when the negation of the theorem statement is easier to be shown to lead an absurd (not true) situation.

To demonstrate the difference between the two proof methods, let us consider the following theorem:

**Theorem 1.** For all integers \( n \), if \( n^2 \) is even then \( n \) is even.

Let \( P \) be the above theorem, then, \( P \) can be written as a quantified predicate as follows.

\[
\forall n \in \mathbb{Z} (p(n) \rightarrow q(n))
\]

where the domain is set of integers, \( p(n) \) is the statement “\( n^2 \) is even” and \( q(n) \) is “\( n \) is even.”

**Proof by contrapositive.** To prove that \( P \) is true by contrapositive, we prove the following statement:

\[
\forall n \in \mathbb{Z} (\neg q(n) \rightarrow \neg p(n))
\]
because if \( p(n) \rightarrow q(n) \) holds for all integers \( n \), then \( \neg q(n) \rightarrow \neg p(n) \) also holds for all integers \( n \).

Now, we have \( \neg q(n) = \text{"n is not even"} \), thus \( n \) is odd, then, \( n = 2k + 1 \) for some integer \( k \) by definition of an odd integer. Then, \( n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \), and we know that \( 2k^2 + 2k \) is also an integer, therefore, \( n^2 \) is also an odd integer. Thus, we showed that “for all integers \( n \), if \( n \) is not even then \( n^2 \) is not even,” hence, we proved that \( \forall n \in \mathbb{Z} (\neg q(n) \rightarrow \neg p(n)) \equiv \forall n \in \mathbb{Z} (p(n) \rightarrow q(n)) \), which is \( P \).

**Proof by contradiction.** To prove that \( P \) is true by contradiction, we assume for contradiction that \( \neg P \) is true. Then, we have

\[
\neg P = \neg \forall n \in \mathbb{Z} (p(n) \rightarrow q(n))
\]

\[
\equiv \exists n \in \mathbb{Z} \neg (p(n) \rightarrow q(n)) \quad \text{By De Morgan’s Law}
\]

\[
\equiv \exists n \in \mathbb{Z} \neg (\neg p(n) \lor q(n)) \quad \text{By Logical Equivalence}
\]

\[
\equiv \exists n \in \mathbb{Z} (\neg \neg p(n) \land \neg q(n)) \quad \text{By De Morgan’s Law}
\]

\[
\equiv \exists n \in \mathbb{Z} (p(n) \land \neg q(n)) \quad \text{By Double Negation Law}
\]

Thus, we are assuming for contradiction that

\[
\neg P = \text{“For some integer \( n \), \( n^2 \) is even but \( n \) is not even.”} \quad (1)
\]

Since \( n \) is not even, \( n \) is odd, which gives us \( n = 2k + 1 \) for some integer \( k \) by definition of an odd integer. Then, \( n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \), and we know that \( 2k^2 + 2k \) is also an integer, therefore, \( n^2 \) is also an odd integer. However, this contradicts that “\( n^2 \) is even” in (1) above, therefore \( \neg P \) is false, hence \( P \), the original theorem statement, is true.