CSCE 222
Discrete Structures for Computing

Recurrence Relations

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Based on slides by Andreas Klappenecker
Modeling with Recurrence Relations
Rabbits (1/3)

[From Leonardo Pisano’s (a.k.a. Fibonacci) book Liber abaci]

A young pair of rabbits, one of each sex, is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month.

Assume that none of the rabbits die.

How many pair of rabbits are there after $n$ months?
Let \( f_n \) denote the number of pairs of rabbits after \( n \) months.

\[
\begin{align*}
  f_1 &= 1 \quad \{\text{reproducing pairs = 0, young pairs = 1}\} \\
  f_2 &= 1 \quad \{\text{reproducing pairs = 0, young pairs = 1}\} \\
  f_3 &= 2 \quad \{\text{reproducing pairs = 1, young pairs = 1}\} \\
  f_4 &= 3 \quad \{\text{reproducing pairs = 1, young pairs = 2}\} \\
  f_5 &= 5 \quad \{\text{reproducing pairs = 2, young pairs = 3}\}
\end{align*}
\]
Rabbits (3/3)

The rabbit population can be modeled by a recurrence relation.

At the end of the first month, the number of pairs of rabbits on the island is \( f_1 = 1 \).

At the end of the second month, the number of pairs of rabbits on the island is \( f_2 = 1 \).

The number of pairs of rabbits after \( n \) months \( f_n \) is equal to the number of pairs of rabbits from the previous month \( f_{n-1} \) plus the number of pairs of newborn rabbits, which equals \( f_{n-2} \), since each newborn pair comes from a pair that is at least two months old, so

\[
f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 3.
\]
Initially, \( n \) discs are placed on the first peg. Move the \( n \) discs one at a time from one peg to another such that no larger disc is ever placed on a smaller disc.

Goal: Move the discs from peg 1 to peg 2.
Let $H_n$ denote the number of moves needed to solve the tower of Hanoi problem with $n$ discs.

1) Move the top $n-1$ discs from peg 1 to peg 3 using $H_{n-1}$ moves.
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2) Move the largest disc from peg 1 to peg 2.
Tower of Hanoi

Let $H_n$ denote the number of moves needed to solve the tower of Hanoi problem with $n$ discs.

1) Move the top $n-1$ discs from peg 1 to peg 3 using $H_{n-1}$ moves.

2) Move the largest disc from peg 1 to peg 2.

3) Move the $n-1$ discs from peg 3 to peg 2 using $H_{n-1}$ moves.
Let $H_n$ denote the number of moves needed to solve the tower of Hanoi problem with $n$ discs.

1) Move the top $n-1$ discs from peg 1 to peg 3 using $H_{n-1}$ moves.
2) Move the largest disc from peg 1 to peg 2.
3) Move the $n-1$ discs from peg 3 to peg 2 using $H_{n-1}$ moves.
Let $H_n$ denote the number of moves needed to solve the tower of Hanoi problem with $n$ discs.

We have $H_n = H_{n-1} + 1 + H_{n-1} = 2H_{n-1} + 1$ for $n \geq 2$,

and $H_1 = 1$. 
Parenthesis

Let $C_n$ denote the number of ways to parenthesize the product of $n+1$ numbers to specify the order of multiplication.

Example: We have $C_2 = 2$ since $(x_0 \cdot x_1) \cdot x_2$ and $x_0 \cdot (x_1 \cdot x_2)$ are the only possibilities to parenthesize three numbers.

Example: We have $C_3 = 5$ since the product $x_0 \cdot x_1 \cdot x_2 \cdot x_3$ can be parenthesized in the following five ways:

- $((x_0 \cdot x_1) \cdot x_2) \cdot x_3$
- $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$
- $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$
- $x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)$
- $x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$
Our goal is to find a recurrence relation for $C_n$.

1) Notice that one multiplication operator remains outside the parentheses, namely the one for the last multiplication to be performed.

2) If the last multiplication appears between $x_k$ and $x_{k+1}$ then there are $C_k C_{n-k-1}$ ways to set the remaining parentheses, as there are $C_k$ ways to multiply the $k+1$ numbers $x_0 \ x_1 \ ... \ x_k$ and the $C_{n-k-1}$ ways to multiply the $n-k$ numbers $x_{k+1} \ ... \ x_n$.

3) Since the final multiplication can appear between any of the $n+1$ numbers, we have $C_n = C_0 \ C_{n-1} + C_1 \ C_{n-2} + ... + C_{n-1} \ C_0$

4) We have $C_0=1$ and $C_1=1$. 
Recurrence Relations
A recurrence system is a finite set of initial conditions

\[ a_0 = C_0, \ a_1 = C_1, \ldots, \ a_k = C_k, \]

where the \( C_0, \ldots, C_k \) are real numbers, and a recurrence relation

\[ a_n = f(a_0, a_1, \ldots, a_{n-1}) \]

expressing \( a_n \) in terms of prior \( a_j \) with \( j < n \).

A sequence \((a_0, a_1, \ldots)\) satisfying the initial condition and the recurrence relation is called a solution.
Example: Tower of Hanoi (1/2)

Recall that the number $H_n$ of moves to solve the Tower of Hanoi puzzle satisfies the recurrence system:

Initial condition: $H_1 = 1$

Recurrence relation: $H_n = 2H_{n-1}+1$ for $n \geq 2$.

For small values of $n$, we get

$(H_1, H_2, H_3, H_4, \ldots) = (1, 3, 7, 15, \ldots)$

Therefore, we can guess that $H_n = 2^n - 1$
How can we prove that $H_n = 2^n - 1$ holds for all $n \geq 1$?

By induction.

Base Step: $H_1 = 1 = 2^1 - 1$, so our claim holds for $n=1$.

Inductive Step: As induction hypothesis (IH), suppose that $H_n = 2^n - 1$ holds. It follows that

$$H_{n+1} = 2H_n + 1 \quad \text{by IH}$$

$$= 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$$

Therefore, the claim follows by induction on $n$. 
The recurrence system

Initial conditions: \( f_0 = 0, f_1 = 1 \)

Recurrence: \( f_n = f_{n-1} + f_{n-2} \) for \( n \geq 2 \).

For small values of \( n \), we get

\[
(f_0, f_1, f_2, \ldots) = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)
\]

We will later learn how to solve it. For any recurrence, we might try the encyclopedia of integer sequences

http://oeis.org
Example: Fibonacci (2/3)

\[ f_n = \frac{1}{2^n \sqrt{5}} \left( \left(1 + \sqrt{5}\right)^n - \left(1 - \sqrt{5}\right)^n \right) \]

The formula might appear mysterious, since we have not yet learned how to derive it. Once we know about generating functions (or characteristic polynomials), it will be a routine matter to find this solution.

From this formula, it is not apparent why \( f_n \) should be an integer. So let’s find out why this must be the case by expanding the formula.
Example: Fibonacci (3/3)

\[ f_n = \frac{1}{2^n \sqrt{5}} \left( \left(1 + \sqrt{5}\right)^n - \left(1 - \sqrt{5}\right)^n \right) \]

\[ = \frac{1}{2^n \sqrt{5}} \left( \sum_{k=0}^{n} \binom{n}{k} \sqrt{5}^k - \sum_{k=0}^{n} \binom{n}{k} (-\sqrt{5})^k \right) \]

\[ = \frac{1}{2^n \sqrt{5}} \sum_{k=0}^{n} 2 \binom{n}{k} \sqrt{5}^k \]

\[ = \frac{1}{2^{n-1} \sqrt{5}} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j+1} \sqrt{5}^{2j+1} \]

\[ = \frac{1}{2^{n-1}} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j+1} \sqrt{5}^{2j} = \frac{1}{2^{n-1}} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j+1} 5^j \]
Example: Parenthesis (1/3)

The number $C_n$ of ways to parenthesize the product of $n+1$ numbers satisfies the recurrence system:

Initial conditions: $C_0=1$ and $C_1=1$

Recurrence relation: $C_n = C_0 C_{n-1} + C_1 C_{n-2} + ... + C_{n-1} C_0$

What is the solution to this recurrence system?
The initial terms of the sequence are

\((C_0, C_1, C_2, C_3, \ldots) = (1,1,2,5,8,14,42,\ldots)\)

Does this ring a bell?

No?

Check the online encyclopedia of integer sequences:

[http://oeis.org](http://oeis.org) (Catalan numbers)
Example: Parenthesis (3/3)

One can show that

\[ C_n = \frac{1}{n + 1} \binom{2n}{n} \]