CSCE 222
Discrete Structures for Computing

Relations

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Based on slides by Andreas Klappenecker
Rabbits

Suppose we have three rabbits called Albert, Bertram, and Chris that have distinct heights.

Let us write \((a,b)\) if \(a\) is taller than \(b\).

Obviously, we cannot have both \((Albert, Bertram)\) and \((Bertram, Albert)\), so not all pairs of rabbit names will occur.

Suppose: Albert is taller than Bertram, and Bertram is taller than Chris.

Then the set of “taller than” relations is:

\[
\{ (Albert, Bertram), (Bertram, Chris), (Albert, Chris) \}
\]
Let $A = \{ \text{Albert, Bertram, Chris} \}$ be the set of rabbits.

Then the “taller than” relations is a subset of the cartesian product $A \times A$, namely $\{(\text{Albert, Bertram}), (\text{Bertram, Chris}), (\text{Albert, Chris})\} \subseteq A \times A$. 
Binary Relations

Let $A$ and $B$ be sets.

A **binary relation** from $A$ to $B$ is a subset of $A \times B$.

A relation **on a set** $A$ is a subset of $A \times A$. 

Examples

Let us consider the following relations on the set of integers:

\[ A = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a \leq b \} \]

\[ B = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a > b \} \]

\[ C = \{ (a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a = b \text{ or } a = -b \} \]
Let $R$ be a relation from $A$ to $B$. In other words, $R$ contains pairs $(a,b)$ with $a$ in $A$ and $b$ in $B$.

If $(a,b)$ in $R$, then we say that $a$ is related to $b$ by $R$.

It is customary to use infix notation for relations.

Thus, we write $a \ R \ b$ to express that $a$ is related to $b$ by $R$. In other words, $a \ R \ b$ iff $(a,b)$ in $R$. 
Example

Let $A$ be the set of city names of the USA. Let $B$ be the set of states. Define the relation $C$

$$C = \{ (a,b) \in A \times B \mid a \text{ is a city of } b \}$$

Then

(College Station, Texas)

(Austin, Texas)

(San Francisco, California)

all belong to the relation $C$. 
The concept of a relation generalizes the concept of a function. A function \( f \) relates the argument \( x \) with its function value \( f(x) \). The difference is that a relation can relate an element \( x \) with more than one value.

For example, consider the relation

\[
A = \{ (a,b) \text{ in } \mathbb{Z} \times \mathbb{Z} \mid a \leq b \}.
\]
We are going to study relations as mathematical objects. This allows us to abstract from well-known relations such as \( \leq \), \( = \), “is taller than”, “likes the same sport as”.

We identify some basic properties of relations. Then we study relations generalizing the equality relation (so-called equivalence relations), and relations generalizing \( \leq \) (so-called partial order relations).
Basic Properties of Relations
We call a relation $R$ on a set $A$ reflexive if and only if $(a,a) \in R$ holds for all $a$ in $A$.

Example: The equality relation $=$ on the set of integers is reflexive, since $a=a$ holds for all integers $a$.

The less than relation $<$ on the set of integers is not reflexive, since $1<1$ does not hold.
X1 Let | denote the divides relation on the set of positive integers, so 2 | 4 means that there exists an integer x such that 2x = 4. Is the relation | reflexive?

X2 Let S be the set of students in this class. Consider the relation R = “wears the same color shirt as.” Is the relation R reflexive?
Symmetry

We call a relation $R$ on a set $A$ **symmetric** if and only if $(a,b) \in R$ implies that $(b,a) \in R$ holds.

Example: The equality relation $=$ on the set of integers is symmetric, since $a=b$ implies that $b=a$.

The less than relation $<$ on the set of integers is not symmetric, since $1<2$ but $2<1$ does not hold.
X1 Let $|$ denote the divides relation on the set of positive integers, so $2 \mid 4$ means that there exists an integer $x$ such that $2x = 4$. Is the relation $|$ symmetric?

X2 Let $S$ be the set of students in this class. Consider the relation $R = \"wears the same color shirt as\"$. Is the relation $R$ symmetric?
We call a relation $R$ on a set $A$ antisymmetric if and only if $(a,b) \in R$ and $(b,a) \in R$ imply that $a=b$.

Formally: $\forall a \forall b ((a,b) \in R \land (b,a) \in R) \rightarrow a=b$.

Example: The equality relation $=$ on the set of integers is antisymmetric, since $a=b$ and $b=a$ implies that $a=b$.

The less than relation $<$ on the set of integers is antisymmetric. Why?
X1 Let $\mid$ denote the divides relation on the set of positive integers, so $2 \mid 4$ means that there exists an integer $x$ such that $2x=4$. Is the relation $\mid$ antisymmetric?

X2 Let $S$ be the set of students in this class. Consider the relation $R = \text{“wears the same color shirt as”}$. Is the relation $R$ antisymmetric?
Warning

The meaning of antisymmetry is not opposite to the meaning of symmetry! In fact, we have already seen that the equality relation $=$ on the set of integers is both symmetric and antisymmetric.

You should very carefully study the meaning of these terms.
Transitive

We call a relation $R$ on a set $A$ **transitive** if and only if $(a,b) \in R$ and $(b,c) \in R$ imply that $(a,c) \in R$.

Example: The equality relation $=$ on the set of integers is transitive, since $a=b$ and $b=c$ implies that $a=c$.

The less than relation $<$ on the set of integers is transitive, since $a<b$ and $b<c$ imply that $a<c$. 
Test Yourself...

X1 Let \( | \) denote the divides relation on the set of positive integers, so \( 2 | 4 \) means that there exists an integer \( x \) such that \( 2x = 4 \). Is the relation \( | \) transitive?

X2 Let \( S \) be the set of students in this class. Consider the relation \( R = \text{“wears the same color shirt as”} \). Is the relation \( R \) transitive?
Equivalence Relations
A relation $R$ on a set $A$ is called an equivalence relation if and only if $R$ is reflexive, symmetric, and transitive.

- Reflexive: For all $a$ in $A$, we have $(a,a)$ in $R$.
- Symmetric: $(a,b)$ in $R$ implies that $(b,a)$ in $R$.
- Transitive: $(a,b)$ in $R$ and $(b,c)$ in $R$ implies that $(a,c)$ in $R$. 
Example: Equality

The equality relation $=$ on the set of integers is an equivalence relation.

Indeed,

the relation $=$ is reflexive, since $a=a$ holds for all integers $a$.

the relation $=$ is symmetric, since $a=b$ implies that $b=a$.

the relation $=$ is transitive, since $a=b$ and $b=c$ implies that $a=c$. 
Example: Congruence mod m

Let m be a positive integer. For integers a and b, we write

\[ a \equiv b \pmod{m} \]

if and only if m divides a-b.

For all \( a \) in \( \mathbb{Z} \), we have \( m \mid (a-a) \), since \( m \cdot 0 = 0 = a-a \). Thus, \( a \equiv a \pmod{m} \) holds for all integers \( a \). Thus, the relation is reflexive.

For \( a, b \) in \( \mathbb{Z} \), if \( a \equiv b \pmod{m} \), then this means that there exists an integer \( k \) such that \( mk = a-b \). Thus, \( m(-k) = b-a \), which implies \( b \equiv a \pmod{m} \). Thus, the relation is symmetric.
Example: Congruence mod m

If \( a \equiv b \pmod{m} \) and \( b \equiv c \pmod{m} \) holds, then this means that there exist integers \( k \) and \( l \) such that

\[
mk = a - b \quad \text{and} \quad ml = b - c
\]

Hence, \( m(k+l) = a - b + b - c = a - c \)

This shows that \( a \equiv c \pmod{m} \) holds.

Therefore, the relation is transitive.

We can conclude that \( a \equiv b \pmod{m} \) is an equivalence relation.
Let $R$ be an equivalence relation on a set $A$. For an element $a$ in $A$, the set of elements

$$[a]_R = \{ b \in A \mid a \, R \, b \}$$

is called the equivalence class of $a$. 
Example

Let us consider the equivalence relation \( a \equiv b \pmod{4} \) on the set of integers. Thus, two integers \( a \) and \( b \) are related whenever their difference is a multiple of 4. Thus, the equivalence classes are:

\[
[0] = \{ ..., -8, -4, 0, 4, 8, ... \}
\]
\[
[1] = \{ ..., -7, -3, 1, 5, 9, ... \}
\]
\[
[2] = \{ ..., -6, -2, 2, 6, ... \}
\]
\[
[3] = \{ ..., -5, -1, 3, 7, ... \}
\]

Now note that \([4] = [0], [5] = [1]\). In fact, \([0], [1], [2], \) and \([3]\) are all equivalence classes.
Theorem

Let $R$ be an equivalence relation on a set $A$. Then the following statements are equivalent:

a) $a \, R \, b$

b) $[a] = [b]$

c) $[a] \cap [b] \neq \emptyset$
Proof

Suppose that $aRb$ holds. We are going to show that $[a] \subseteq [b]$ holds. Let $c \in [a]$. This means that $aRc$ holds. Since $R$ is symmetric, $aRb$ implies that $bRa$. By transitivity, $bRa$ and $aRc$ imply that $bRc$ holds. Hence, $c \in [b]$. Therefore, we have shown that $[a] \subseteq [b]$. The proof that $[b] \subseteq [a]$ is similar. Hence, we have shown that statement a) implies statement b).

We will show now that b) implies c). Since $a \in [a]$, we know that the equivalence class of $a$ is not empty. As $[a] = [b] \neq \emptyset$, we have $[a] \cap [b] \neq \emptyset$. 
Proof (continued)

We will show now that c) implies a).

Suppose that \([a] \cap [b] \neq \emptyset\). Thus, there exists an element \(c\) such that \(aRc\) and \(bRc\). By symmetry, we get \(cRb\). It follows by transitivity that \(aRb\) holds. \(\square\)
Partial Order Relations
A relation $R$ on a set $A$ is called a **partial order** if and only if it is reflexive, antisymmetric, and transitive.

A set $A$ with a partial order is called a partially ordered set (poset).
Example 1

The “less than or equal to” relation \( \leq \) on the set of integers is a partial order relation.

Indeed, since \( a \leq a \) holds for all integers \( a \), the relation \( \leq \) is reflexive.

Since \( a \leq b \) and \( b \leq a \) implies that \( a = b \), the relation is antisymmetric.

Since \( a \leq b \) and \( b \leq c \) implies that \( a \leq c \), the relation is transitive.
Example 2

The divides relation | on the set of positive integers is a partial order relation.

Indeed, since a|a for all positive integers a, the relation | is reflexive.

If a|b and b|a, then there exist integers k and l such that a k = b and b l = a. Therefore, a kl = a, so kl=1. This means that either k=l=1 or k=l=-1. Since a and b are positive integers, we cannot have a (-1) = b. Therefore, we must have k=l=1, which means that a=b. Thus, | is an antisymmetric relation.

The relation | is transitive, since a|b and b|c means that there exist integers k and l such that ak=b and bl=c, so a(kl)=c, which implies that a | c.
X1 Is the less than relation $<$ on the set of integers a partial order relation?

X2 Let $S$ be a set. Is the subset relation $\subseteq$ on the set $P(S)$ a partial order relation?
Comparable Elements

A partial order on a set $S$ is often denoted by symbols resembling the notation commonly used for “less than or equal to”, namely $\leq$ or $\subseteq$ or $\leq$.

Let $(S, \leq)$ be a partially ordered set. For two elements $a$ and $b$ of $S$, we do not necessarily have that one of the relations $a \leq b$ or $b \leq a$ holds. If one of them holds, then we call $a$ and $b$ comparable elements of $S$, otherwise $a$ and $b$ are incomparable.
A partially ordered set $(S, \leq)$ in which any two elements are comparable is called a total order.

A totally ordered set is also called a chain.

For example, consider the set of positive integers $\mathbb{N}$ with $\leq$. Any two positive integers are comparable with $\leq$. It can form a chain such that $1 \leq 2 \leq 3 \leq 4 \leq \ldots$
Suppose that we have two partially ordered sets: 

\((A, \leq_1)\) and \((B, \leq_2)\).

We can construct a partial order on \(A \times B\) by defining 

\[(a_1, b_1) \leq (a_2, b_2)\]

if and only if \((a_1 = a_2\) and \(b_1 \leq_2 b_2\)) or \((a_1 <_1 a_2)\) holds.

We call the relation \(\leq\) the **lexicographic order** on the cartesian product \(A \times B\).
Example

Let $\mathbb{Z}$ be the set of integers, totally ordered with the “less than or equal to” relation $\leq$.

In the lexicographic order $\preceq$ on $\mathbb{Z} \times \mathbb{Z}$, we have

$(3,4) \preceq (4,2)$

$(3,7) \preceq (3,8)$
Let \((S, \leq)\) be a finite partially ordered set.

Suppose that \(a\) and \(b\) are distinct elements of \(S\) such that \(a \leq b\). We say that \(b\) covers \(a\) if and only if there does not exist an element \(c\) in \(S\) such that \(a < c < b\).

The Hasse diagram of \((S, \leq)\) is a diagram in which an element \(b\) of \(S\) is written above \(a\) and connected by a line if and only if \(b\) covers \(a\).
Examples

Consider \{2,4,5,10,12,20,25\} with divisibility condition.

The Hasse diagram is given by

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet \\
12 & 20 & 4 & 10 & 25 \\
\end{array}
\]

The cover relation for this Hasse diagram is \{(2,4), (2,10), (4,12), (4,20), (5,10), (5,25), (10,20)\}.

The Hasse diagram is more economical than representing the partial order relation by a directed graph (with an edge from a to b whenever a <= b). Self-loops and transitively implied relations are omitted.
Maximal and Minimal Elements

Let \((S, \leq)\) be a partially ordered set.

An element \(m\) in \(S\) is called \textit{maximal} iff there does not exist any element \(b\) in \(S\) such that \(m < b\).

An element \(m\) in \(S\) is called \textit{minimal} iff there does not exist any element \(b\) in \(S\) such that \(b < m\).
Example

Determine the maximal elements of the set
\(\{2, 4, 5, 10, 12, 20, 25\}\),
partially ordered by the divisibility relation.
The elements 12, 20, and 25 are the maximal elements.

Determine the minimal elements of the above partially ordered set \((\{2, 4, 5, 10, 12, 20, 25\}, \mid)\).
The elements 2 and 5 are the minimal elements.
Least and Greatest Element

Let \((S, \leq)\) be a partially ordered set.

An element \(a\) in \(S\) is called the least element iff \(a \leq b\) holds for all \(b\) in \(S\).

[A least element does not need to exist. If it does, then it is uniquely determined.]

An element \(z\) in \(S\) is called the greatest element iff \(b \leq z\) holds for all \(b\) in \(S\).

[A greatest element does not need to exist. If it does, then it is uniquely determined.]
X1 Determine the least and the greatest elements of the set of positive integers partially ordered by divisibility.

X2 Let $S$ be a nonempty set. Partially order the power set $P(S)$ by inclusion. Determine the least and the greatest elements of $P(S)$. 
Lattices
Upper and Lower Bounds

Let \((S, \leq)\) be a partially ordered set.

Let \(A\) be a subset of \(S\).

An element \(u\) of \(S\) is called an **upper bound** of \(A\) if and only if \(a \leq u\) holds for all \(a\) in \(A\).

An element \(l\) of \(S\) is called a **lower bound** of \(A\) if and only if \(l \leq a\) holds for all \(a\) in \(A\).
Example

Let \(\{2, 4, 5, 10, 12, 20, 25\}\) with divisibility condition.

The Hasse diagram is given by

The subset \(A = \{4, 10\}\) has 20 as an upper bound, and 2 as a lower bound.

The subset \(A = \{12\}\) has 12 as an upper bound, and 2, 4 and 12 as lower bounds.
Least Upper Bounds

Let \((S, \leq)\) be a partially ordered set, and \(A\) a subset of \(S\). An element \(u\) of \(S\) is called a **least upper bound** of \(A\) if it is an upper bound that is less than any other upper bound of \(A\).

[Unlike upper bounds, the least upper bound is uniquely determined if it exists]
Greatest Lower Bounds

Let \((S, \leq)\) be a partially ordered set, and \(A\) a subset of \(S\). An element \(l\) of \(S\) is called a greatest lower bound of \(A\) if it is a lower bound that is greater than any other lower bound of \(A\).

A greatest lower bound is uniquely determined if it exists.
Consider the poset \((S=\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, | )\).

Draw the Hasse diagram:

What are the upper bounds of the subset \(A=\{2, 9\}\)?

What are the lower bounds of the subset \(B=\{60, 72\}\)?
Lattices

A partially ordered set in which every pair has both a least upper bound and a greatest lower bound is called a lattice.
Example

Consider the set \((\mathbb{N},|)\) of positive integers that is partially ordered with respect to the divisibility relation.

Let \(a\) and \(b\) be two distinct positive integers. Then the least upper bound of \(\{a,b\}\) is the least common multiple of \(a\) and \(b\). The greatest lower bound is the greatest common divisor of \(\{a,b\}\). Therefore, \((\mathbb{N},|)\) is a lattice.