CS545—Contents VII

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whereas the other equivalent solution for ϑ in the range $(\pi/2, 3\pi/2)$ is

$$\begin{aligned}\varphi &= \text{Atan2}(-r_{21}, -r_{11})\\ \vartheta &= \text{Atan2}\left(-r_{31}, -\sqrt{r_{32}^2 + r_{33}^2}\right)\\ \psi &= \text{Atan2}(-r_{32}, -r_{33}).\end{aligned} \tag{2.26'}$$

Solutions (2.26) and (2.26′) degenerate when $c_\vartheta = 0$; in this case, it is possible to determine only the sum or difference of φ and ψ.

2.6 HOMOGENEOUS TRANSFORMATIONS

As illustrated at the beginning of the chapter, the position of a rigid body in space is expressed in terms of the position of a suitable point on the body with respect to a reference frame (translation), while its orientation is expressed in terms of the components of the unit vectors of a frame attached to the body—with origin in the above point—with respect to the same reference frame (rotation).

As shown in Fig. 2.11, consider an arbitrary point P in space. Let $\boldsymbol{p}^0$ be the vector of coordinates of P with respect to the reference frame O_0–$x_0y_0z_0$. Consider then another frame in space O_1–$x_1y_1z_1$. Let $\boldsymbol{o}_1^0$ be the vector describing the origin of frame 1 with respect to frame 0, and $\boldsymbol{R}_1^0$ be the rotation matrix of frame 1 with respect to frame 0. Let also $\boldsymbol{p}^1$ be the vector of coordinates of P with respect to frame 1. On the basis of simple geometry, the position of point P with respect to the reference frame can be expressed as

$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0\boldsymbol{p}^1. \tag{2.27}$$

Hence, Eq. (2.27) represents the *coordinate transformation* (*translation* + *rotation*) of a bound vector between two frames.

The inverse transformation can be obtained by premultiplying both sides of (2.27) by $\boldsymbol{R}_1^{0T}$; in view of (2.4), it follows that

$$\boldsymbol{p}^1 = -\boldsymbol{R}_1^{0T}\boldsymbol{o}_1^0 + \boldsymbol{R}_1^{0T}\boldsymbol{p}^0 \tag{2.28}$$

which, via (2.16), can be written as

$$\boldsymbol{p}^1 = -\boldsymbol{R}_0^1\boldsymbol{o}_1^0 + \boldsymbol{R}_0^1\boldsymbol{p}^0. \tag{2.29}$$

In order to achieve a compact representation of the relationship between the coordinates of the same point in two different frames, the *homogeneous representation* of a generic vector $\boldsymbol{p}$ can be introduced as the vector $\tilde{\boldsymbol{p}}$ formed by adding a fourth unit component:

$$\tilde{\boldsymbol{p}} = \begin{bmatrix}\boldsymbol{p}\\ 1\end{bmatrix}. \tag{2.30}$$

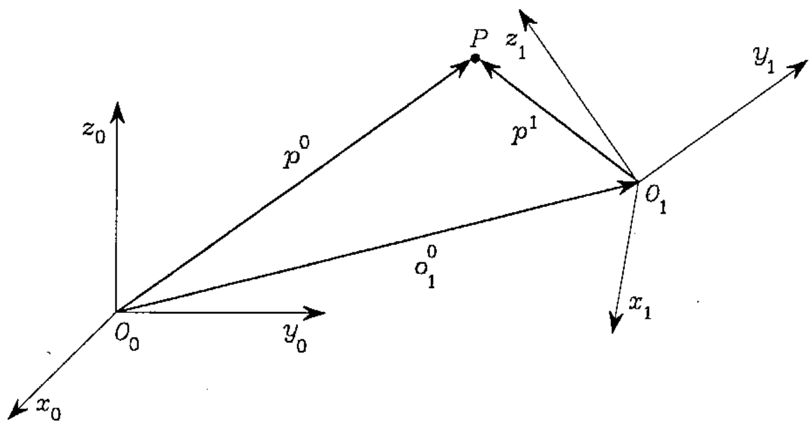

FIGURE 2.11
Representation of a point P in different coordinate frames.

By adopting this representation for the vectors $\boldsymbol{p}^0$ and $\boldsymbol{p}^1$ in (2.27), the coordinate transformation can be written in terms of the (4×4) matrix

$$\boldsymbol{A}_1^0 = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix} \tag{2.31}$$

which, according to (2.30), is termed *homogeneous transformation matrix.* As can be seen from (2.31), the transformation of a vector from frame 1 to frame 0 is expressed by a single matrix containing the rotation matrix of frame 1 with respect to frame 0 and the translation vector from the origin of frame 0 to the origin of frame 1[4]. Therefore, the coordinate transformation (2.27) can be compactly rewritten as

$$\tilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \tilde{\boldsymbol{p}}^1. \tag{2.32}$$

Coordinate transformation between frame 0 and frame 1 is described by the homogeneous transformation matrix $\boldsymbol{A}_0^1$ which satisfies the equation

$$\tilde{\boldsymbol{p}}^1 = \boldsymbol{A}_0^1 \tilde{\boldsymbol{p}}^0 = \left(\boldsymbol{A}_1^0\right)^{-1} \tilde{\boldsymbol{p}}^0. \tag{2.33}$$

This matrix is expressed in a block-partitioned form as

$$\boldsymbol{A}_0^1 = \begin{bmatrix} \boldsymbol{R}_1^{0T} & -\boldsymbol{R}_1^{0T}\boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_0^1 & -\boldsymbol{R}_0^1\boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}, \tag{2.34}$$

[4] It can be shown that in (2.31) nonnull values of the first three elements of the fourth row of $\boldsymbol{A}$ produce a perspective effect, while values other than unity for the fourth element give a scaling effect.

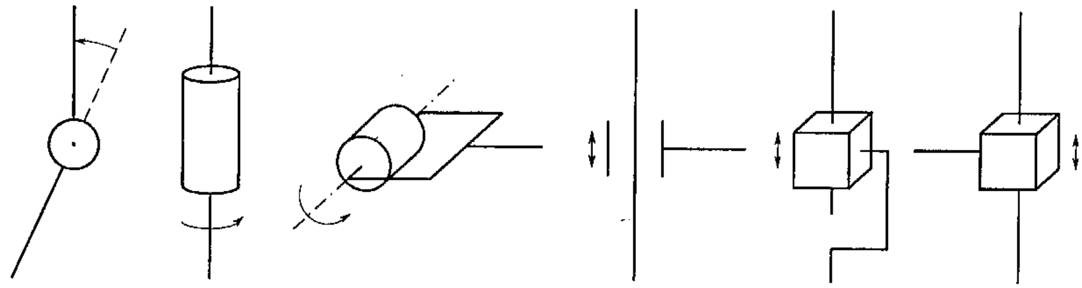

FIGURE 2.12
Conventional representations of joints.

which gives the homogeneous representation form of the result already established by (2.28) and (2.29).

Notice that for the homogeneous transformation matrix the orthogonality property does not hold; hence, in general,

$$A^{-1} \neq A^T. \tag{2.35}$$

In sum, a homogeneous transformation matrix expresses the coordinate transformation between two frames in a compact form. If the frames have the same origin, it reduces to the rotation matrix previously defined. Instead, if the frames have distinct origins, it allows keeping the notation with superscripts and subscripts that directly characterize the current frame and the fixed frame.

Analogously to what presented for the rotation matrices, it is easy to verify that a sequence of coordinate transformations can be composed by the product

$$\tilde{p}^0 = A_1^0 A_2^1 \dots A_n^{n-1} \tilde{p}^n \tag{2.36}$$

where A_i^{i-1} denotes the homogeneous transformation relating the description of a point in frame i to the description of the same point in frame $i-1$.

2.7 DIRECT KINEMATICS

A manipulator consists of a series of rigid bodies (*links*) connected by means of kinematic pairs or *joints*. It is assumed that each joint provides the mechanical structure with a single degree of mobility, corresponding to the articulation or *joint variable*. Joints can essentially be of two types: *revolute* and *prismatic*; conventional representations of the two types of joints are sketched in Fig. 2.12. The whole structure forms an *open kinematic chain*[5]. One end of the chain is constrained to a base. An *end effector* (gripper, tool) is connected to the other end allowing manipulation of objects in space.

[5] The open kinematic chain is the fundamental structure of an industrial robot, although there exist industrial manipulators having closed kinematic chains.

CHAPTER 2

KINEMATICS

A *manipulator* can be schematically represented from a mechanical viewpoint as an open kinematic chain of rigid bodies (*links*) connected by means of revolute or prismatic *joints* which constitute the degrees of mobility of the structure. One end of the chain is constrained to a base, while an *end effector* is mounted on the other end. The resulting motion of the structure is obtained by composition of the elementary motions of each link with respect to the previous one. Therefore, in order to manipulate an object in space, it is necessary to describe the end-effector position and orientation. This chapter is dedicated to the derivation of the *direct kinematics equation* through a systematic, general approach based on linear algebra. This allows expressing the end-effector position and orientation as a function of the joint variables of the mechanical structure with respect to a reference frame. With reference to a *minimal representation of orientation*, the concept of *operational space* is introduced and its relationship with the *joint space* is established. Furthermore, a *calibration* technique of the manipulator kinematic parameters is presented. The chapter ends with the derivation of solutions to the *inverse kinematics problem*, which consists of the determination of the joint variables corresponding to a given end-effector configuration.

2.1 POSITION AND ORIENTATION OF A RIGID BODY

A *rigid body* is completely described in space by its *position* and *orientation* with respect to a reference frame. As shown in Fig. 2.1, let O–xyz be the orthonormal reference frame and $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$ be the unit vectors of the frame axes.

The position of a point O' on the rigid body with respect to the coordinate frame O–xyz is expressed by the relation

$$\boldsymbol{o}' = o'_x \boldsymbol{x} + o'_y \boldsymbol{y} + o'_z \boldsymbol{z},$$

where o'_x, o'_y, o'_z denote the components of the vector $\boldsymbol{o}'$ along the frame axes; the position of O' can be compactly written as the (3×1) vector

$$\boldsymbol{o}' = \begin{bmatrix} o'_x \\ o'_y \\ o'_z \end{bmatrix}. \tag{2.1}$$

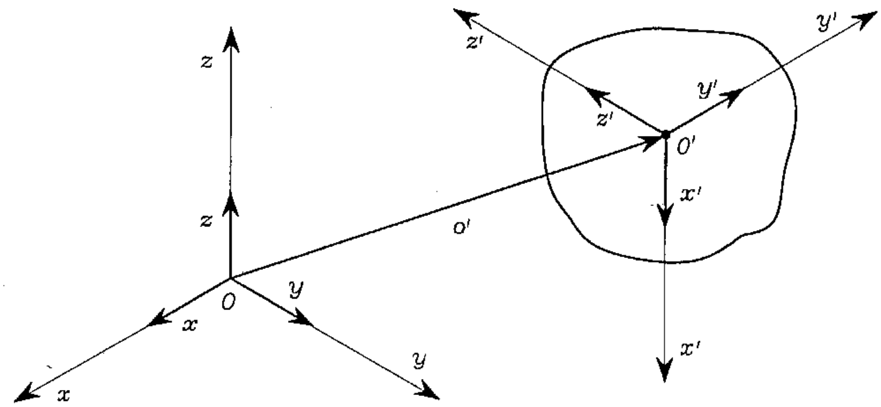

FIGURE 2.1
Position and orientation of a rigid body.

Vector $\boldsymbol{o}'$ is a bound vector since its line of application and point of application are both prescribed, in addition to its direction and norm.

In order to describe the rigid body orientation, it is convenient to consider an orthonormal frame attached to the body and express its unit vectors with respect to the reference frame. Let then O'–$x'y'z'$ be such frame with origin in O' and $\boldsymbol{x}'$, $\boldsymbol{y}'$, $\boldsymbol{z}'$ be the unit vectors of the frame axes. These vectors are expressed with respect to the reference frame O–xyz by the equations:

$$\begin{aligned} \boldsymbol{x}' &= x'_x \boldsymbol{x} + x'_y \boldsymbol{y} + x'_z \boldsymbol{z} \\ \boldsymbol{y}' &= y'_x \boldsymbol{x} + y'_y \boldsymbol{y} + y'_z \boldsymbol{z} \\ \boldsymbol{z}' &= z'_x \boldsymbol{x} + z'_y \boldsymbol{y} + z'_z \boldsymbol{z}. \end{aligned} \tag{2.2}$$

The components of each unit vector are the direction cosines of the axes of frame O'–$x'y'z'$ with respect to the reference frame O–xyz.

2.2 ROTATION MATRIX

By adopting a compact notation, the three unit vectors in (2.2) describing the body orientation with respect to the reference frame can be combined in the (3×3) matrix

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}, \tag{2.3}$$

which is termed *rotation matrix.*

It is worth noting that the column vectors of matrix $\boldsymbol{R}$ are mutually orthogonal since they represent the unit vectors of an orthonormal frame, i.e.,

$$\boldsymbol{x}'^T\boldsymbol{y}' = 0 \qquad \boldsymbol{y}'^T\boldsymbol{z}' = 0 \qquad \boldsymbol{z}'^T\boldsymbol{x}' = 0.$$

Also, they have unit norm

$$x'^T x' = 1 \qquad y'^T y' = 1 \qquad z'^T z' = 1.$$

As a consequence, $\boldsymbol{R}$ is an *orthogonal* matrix meaning that

$$\boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I} \tag{2.4}$$

where $\boldsymbol{I}$ denotes the (3×3) identity matrix.

If both sides of (2.4) are postmultiplied by the inverse matrix $\boldsymbol{R}^{-1}$, the useful result is obtained:

$$\boldsymbol{R}^T = \boldsymbol{R}^{-1}, \tag{2.5}$$

that is, the transpose of the rotation matrix is equal to its inverse. Further, observe that $\det(\boldsymbol{R}) = 1$ if the frame is right-handed, while $\det(\boldsymbol{R}) = -1$ if the frame is left-handed.

2.2.1 Elementary Rotations

Consider the frames that can be obtained via *elementary rotations* of the reference frame about one of the coordinate axes. These rotations are positive if they are made counter-clockwise about the relative axis.

Suppose that the reference frame O–xyz is rotated by an angle α about axis z (Fig. 2.2), and let O–$x'y'z'$ be the rotated frame. The unit vectors of the new frame can be described in terms of their components with respect to the reference frame, i.e.,

$$x' = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \qquad y' = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \qquad z' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, the rotation matrix of frame O–$x'y'z'$ with respect to frame O–xyz is

$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.6}$$

In a similar manner, it can be shown that the rotations by an angle β about axis y and by an angle γ about axis x are respectively given by:

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{2.7}$$

$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}. \tag{2.8}$$

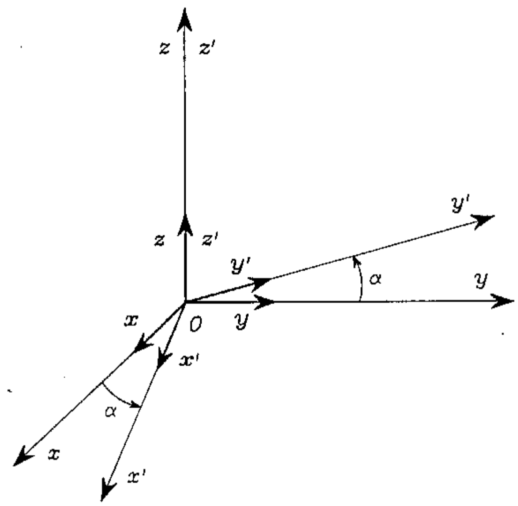

FIGURE 2.2
Rotation of the frame O–xyz by an angle α about axis z.

These matrices will be useful to describe rotations about an arbitrary axis in space.

It is easy to verify that for the elementary rotation matrices in (2.6)–(2.8) the following property holds:

$$\boldsymbol{R}_k(-\vartheta) = \boldsymbol{R}_k^T(\vartheta) \qquad\qquad k = x, y, z. \tag{2.9}$$

Relations (2.6)–(2.8) allow attributing a geometrical meaning to the rotation matrix; the matrix $\boldsymbol{R}$ describes the rotation about an axis in space needed to align the axes of the reference frame with the corresponding axes of the body frame.

2.2.2 Representation of a Vector

In order to understand a further geometrical meaning of a rotation matrix, consider the case when the origin of the body frame coincides with the origin of the reference frame (Fig. 2.3); it follows that $\boldsymbol{o}' = \boldsymbol{0}$, where $\boldsymbol{0}$ denotes the (3×1) null vector. A point P in space can be represented either as

$$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

with respect to frame O–xyz, or as

$$\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

with respect to frame O–$x'y'z'$.

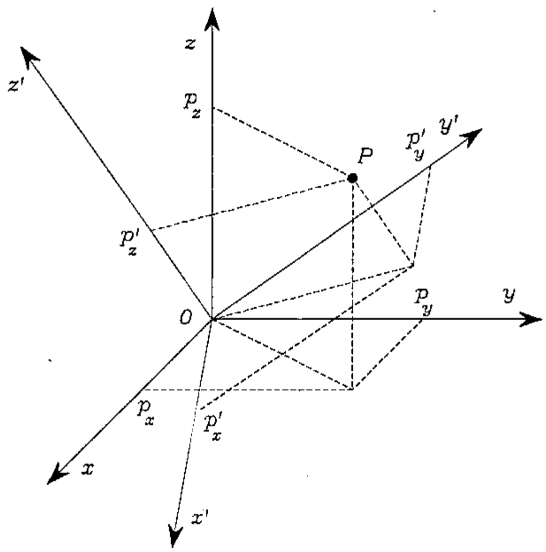

FIGURE 2.3
Representation of a point P in two different coordinate frames.

Since $\boldsymbol{p}$ and $\boldsymbol{p}'$ are representations of the same point P, it is

$$\boldsymbol{p} = p'_x \boldsymbol{x}' + p'_y \boldsymbol{y}' + p'_z \boldsymbol{z}' = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} \boldsymbol{p}'$$

and, accounting for (2.3), it is

$$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'. \tag{2.10}$$

The rotation matrix $\boldsymbol{R}$ represents the *transformation matrix* of the vector coordinates in frame $O\text{–}x'y'z'$ into the coordinates of the same vector in frame $O\text{–}xyz$. In view of the orthogonality property (2.4), the inverse transformation is simply given by

$$\boldsymbol{p}' = \boldsymbol{R}^T \boldsymbol{p}. \tag{2.11}$$

Example 2.1. Consider two frames with common origin mutually rotated by an angle α about the axis z. Let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be the vectors of the coordinates of a point P, expressed in the frames $O\text{–}xyz$ and $O\text{–}x'y'z'$, respectively (Fig. 2.4). On the basis of simple geometry, the relationship between the coordinates of P in the two frames is:

$$\begin{aligned} p_x &= p'_x \cos\alpha - p'_y \sin\alpha \\ p_y &= p'_x \sin\alpha + p'_y \cos\alpha \\ p_z &= p'_z. \end{aligned}$$

Therefore, the matrix (2.6) represents not only the orientation of a frame with respect to another frame, but it also describes the transformation of a vector from a frame to another frame with the same origin.

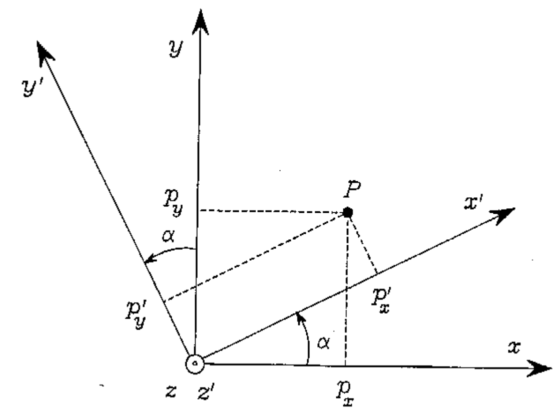

FIGURE 2.4
Representation of a point P in rotated frames.

2.2.3 Rotation of a Vector

A rotation matrix can be also interpreted as the matrix operator allowing rotation of a vector by a given angle about an arbitrary axis in space. In fact, let $\boldsymbol{p}'$ be a vector in the reference frame O–xyz; in view of orthogonality of the matrix $\boldsymbol{R}$, the product $\boldsymbol{R}\boldsymbol{p}'$ yields a vector $\boldsymbol{p}$ with the same norm as that of $\boldsymbol{p}'$ but rotated with respect to $\boldsymbol{p}'$ according to the direction and rotation angle established by $\boldsymbol{R}$. The norm equality can be proved by observing that $\boldsymbol{p}^T\boldsymbol{p} = \boldsymbol{p}'^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{p}'$ and applying (2.4). This interpretation of the rotation matrix will be revisited later.

Example 2.2. Consider the vector $\boldsymbol{p}$ which is obtained by rotating a vector $\boldsymbol{p}'$ in the plane xy by an angle α about axis z of the reference frame (Fig. 2.5). Let (p'_x, p'_y, p'_z) be the coordinates of the vector $\boldsymbol{p}'$. The vector $\boldsymbol{p}$ has components

$$p_x = p'_x \cos\alpha - p'_y \sin\alpha$$
$$p_y = p'_x \sin\alpha + p'_y \cos\alpha$$
$$p_z = p'_z.$$

It is easy to recognize that $\boldsymbol{p}$ can be expressed as

$$\boldsymbol{p} = \boldsymbol{R}_z(\alpha)\boldsymbol{p}',$$

where $\boldsymbol{R}_z(\alpha)$ is the same rotation matrix as in (2.6).

In sum, a rotation matrix attains three *equivalent geometrical meanings*:

- It describes the mutual orientation between two coordinate frames; its column vectors are the direction cosines of the axes of the rotated frame with respect to the original frame.
- It represents the coordinate transformation between the coordinates of a point expressed in two different frames (with common origin).
- It is the operator that allows rotating a vector in the same coordinate frame.

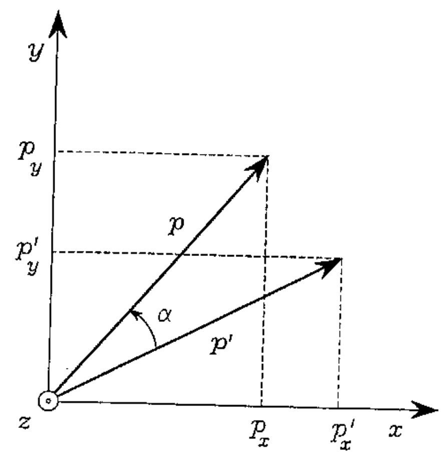

FIGURE 2.5
Rotation of a vector.

2.3 COMPOSITION OF ROTATION MATRICES

In order to derive composition rules of rotation matrices, it is useful to consider the expression of a vector in two different reference frames. Let then O–$x_0y_0z_0$, O–$x_1y_1z_1$, and O–$x_2y_2z_2$ be three frames with common origin O. The vector $\boldsymbol{p}$ describing the position of a generic point in space can be expressed in each of the above frames; let $\boldsymbol{p}^0$, $\boldsymbol{p}^1$, and $\boldsymbol{p}^2$ denote the expressions of $\boldsymbol{p}$ in the three frames[1].

At first, consider the relationship between the expression $\boldsymbol{p}^2$ of the vector $\boldsymbol{p}$ in frame 2 and the expression $\boldsymbol{p}^1$ of the same vector in frame 1. If $\boldsymbol{R}_i^j$ denotes the rotation matrix of frame i with respect to frame j, it is

$$\boldsymbol{p}^1 = \boldsymbol{R}_2^1\boldsymbol{p}^2. \tag{2.12}$$

Similarly, it turns out that

$$\boldsymbol{p}^0 = \boldsymbol{R}_1^0\boldsymbol{p}^1 \tag{2.13}$$
$$\boldsymbol{p}^0 = \boldsymbol{R}_2^0\boldsymbol{p}^2. \tag{2.14}$$

On the other hand, substituting (2.12) in (2.13) and using (2.14) gives

$$\boldsymbol{R}_2^0 = \boldsymbol{R}_1^0\boldsymbol{R}_2^1. \tag{2.15}$$

Equation (2.15) can be interpreted as the composition of successive rotations. Consider a frame initially aligned with the frame O–$x_0y_0z_0$. The rotation expressed by matrix $\boldsymbol{R}_2^0$ can be regarded as obtained in two steps:

[1] In the remainder, the superscript of a vector or a matrix denotes the frame in which its components are expressed.

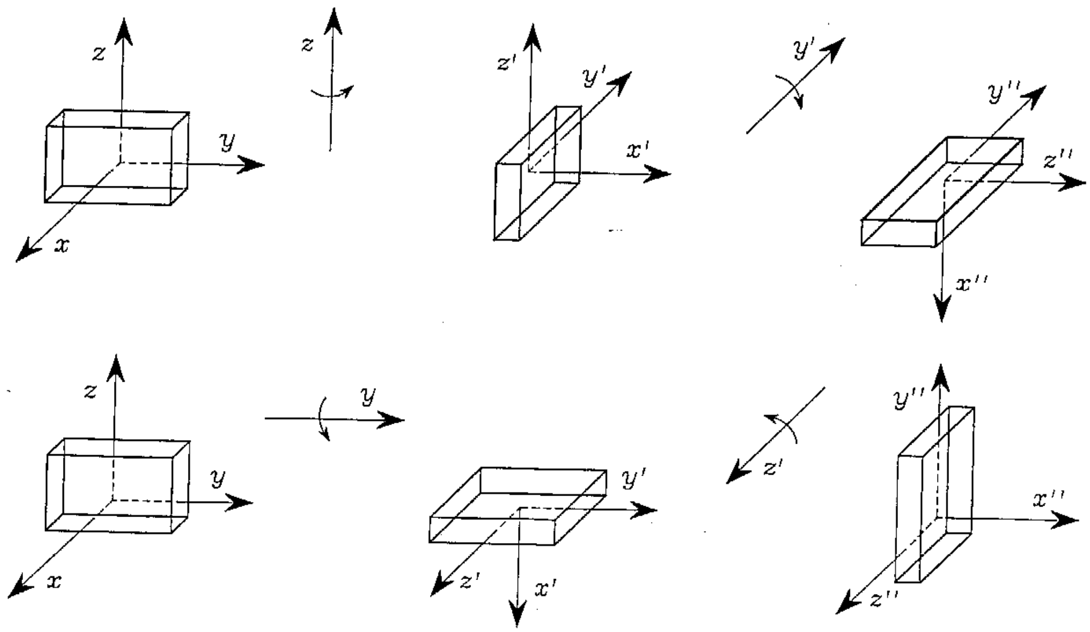

FIGURE 2.6
Successive rotations of an object about axes of current frame.

- first rotate the given frame according to $\boldsymbol{R}_1^0$, so as to align it with frame $O\text{–}x_1y_1z_1$;
- then rotate the frame, now aligned with frame $O\text{–}x_1y_1z_1$, according to $\boldsymbol{R}_2^1$, so as to align it with frame $O\text{–}x_2y_2z_2$.

Notice that the overall rotation can be expressed as a sequence of partial rotations; each rotation is defined with respect to the preceding one. The frame with respect to which the rotation occurs is termed *current frame*. Composition of successive rotations is then obtained by postmultiplication of the rotation matrices following the given order of rotations, as in (2.15). With the adopted notation, in view of (2.5), it is

$$\boldsymbol{R}_i^j = (\boldsymbol{R}_j^i)^{-1} = (\boldsymbol{R}_j^i)^T. \tag{2.16}$$

Successive rotations can be also specified by constantly referring them to the initial frame; in this case, the rotations are made with respect to a *fixed frame*. Let $\boldsymbol{R}_1^0$ be the rotation matrix of frame $O\text{–}x_1y_1z_1$ with respect to the fixed frame $O\text{–}x_0y_0z_0$. Let then $\boldsymbol{R}_2^0$ denote the matrix characterizing frame $O\text{–}x_2y_2z_2$ with respect to frame 0, which is obtained as a rotation of frame 1 according to the matrix $\boldsymbol{R}_2^1$. Since Eq. (2.15) gives a composition rule of successive rotations about the axes of the current frame, the overall rotation can be regarded as obtained in the following steps:

- first realign frame 1 with frame 0 by means of rotation $\boldsymbol{R}_0^1$;
- then make the rotation expressed by $\boldsymbol{R}_2^1$ with respect to the current frame;
- finally compensate for the rotation made for the realignment by means of the inverse rotation $\boldsymbol{R}_1^0$.

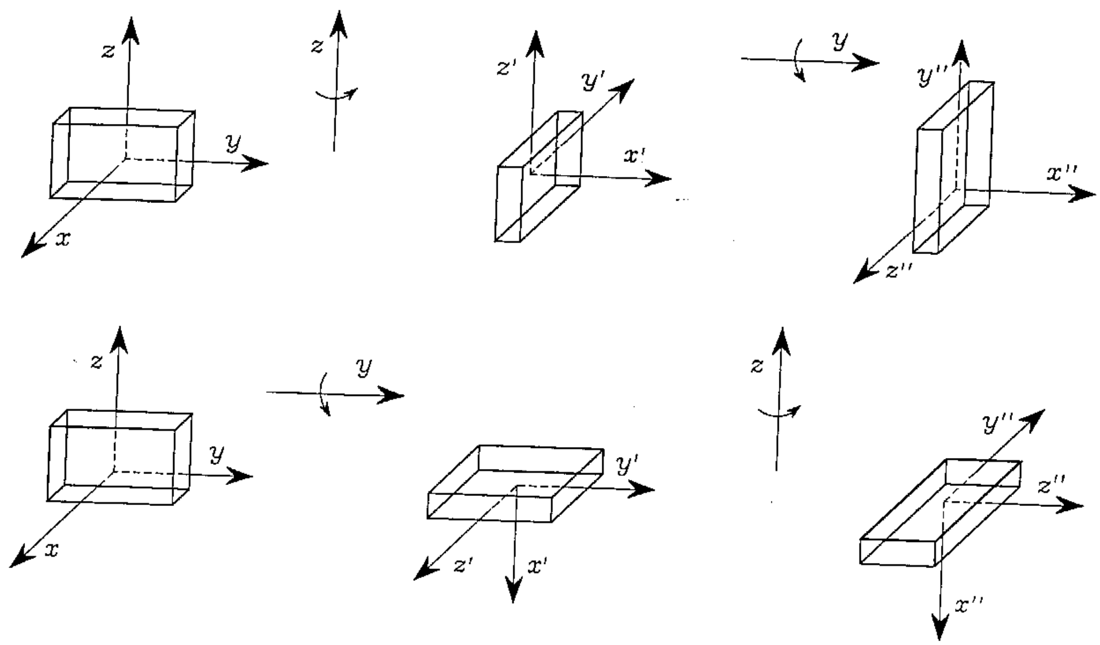

FIGURE 2.7
Successive rotations of an object about axes of fixed frame.

Since the above rotations are described with respect to the current frame, application of the composition rule (2.15) yields

$$\boldsymbol{R}_2^0 = \boldsymbol{R}_1^0 \boldsymbol{R}_0^1 \boldsymbol{R}_2^1 \boldsymbol{R}_1^0.$$

In view of (2.16), it is

$$\boldsymbol{R}_2^0 = \boldsymbol{R}_2^1 \boldsymbol{R}_1^0 \tag{2.17}$$

where the resulting $\boldsymbol{R}_2^0$ is different from the matrix in (2.15). Hence, it can be stated that composition of successive rotations with respect to a fixed frame is obtained by premultiplication of the single rotation matrices in the order of the given sequence of rotations.

By recalling the meaning of a rotation matrix in terms of the orientation of a current frame with respect to a fixed frame, it can be recognized that its columns are the direction cosines of the axes of the current frame with respect to the fixed frame, while its rows (columns of its transpose and inverse) are the direction cosines of the axes of the fixed frame with respect to the current frame.

An important issue of composition of rotations is that the matrix product is not commutative. In view of this, it can be concluded that two rotations in general do not commute and its composition depends on the order of the single rotations.

Example 2.3. Consider an object and a frame attached to it. Fig. 2.6 shows the effects of two successive rotations of the object with respect to the current frame by changing the order of rotations. It is evident that the final object orientation is different in the two cases. Also in the case of rotations made with respect to the current frame, the final orientations differ (Fig. 2.7). It is interesting to note that the effects of the sequence of rotations with respect to the fixed frame are interchanged with the effects of the sequence of rotations with respect to the current frame. This can be explained by observing that the order of rotations in the fixed frame commutes with respect to the order of rotations in the current frame.

2.4 ROTATION ABOUT AN ARBITRARY AXIS

It is often desired to express a rotation of a given angle about an arbitrary axis in space. This can be advantageous in the problem of trajectory planning for a manipulator's end-effector orientation.

Let $\boldsymbol{r} = [\, r_x \quad r_y \quad r_z \,]^T$ be the unit vector of a rotation axis with respect to the reference frame O–xyz. In order to derive the rotation matrix $\boldsymbol{R}_r(\vartheta)$ expressing the rotation of an angle ϑ about axis $\boldsymbol{r}$, it is convenient to compose elementary rotations about the coordinate axes of the reference frame. The angle is taken to be positive if the rotation is made counter-clockwise about axis $\boldsymbol{r}$.

As shown in Fig. 2.8, a possible solution is to rotate first $\boldsymbol{r}$ by the angle necessary to align it with axis z, then to rotate by ϑ about z and finally to rotate by the angle necessary to align the unit vector with the initial direction. In detail, the sequence of rotations, to be made always with respect to axes of fixed frame, is the following:

- align $\boldsymbol{r}$ with z, which is obtained as the sequence of a rotation by $-\alpha$ about z and a rotation by $-\beta$ about y;
- rotate by ϑ about z;
- realign with the initial direction of $\boldsymbol{r}$, which is obtained as the sequence of a rotation by β about y and a rotation by α about z.

In sum, the resulting rotation matrix is

$$\boldsymbol{R}_r(\vartheta) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\vartheta)\boldsymbol{R}_y(-\beta)\boldsymbol{R}_z(-\alpha). \tag{2.18}$$

From the components of the unit vector $\boldsymbol{r}$ it is possible to extract the transcendental functions needed to compute the rotation matrix in (2.18), so as to eliminate the dependence from α and β; in fact, it is

$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2 + r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin\beta = \sqrt{r_x^2 + r_y^2} \qquad \cos\beta = r_z.$$

Then, it can be found that the rotation matrix expressing the *rotation about an arbitrary*

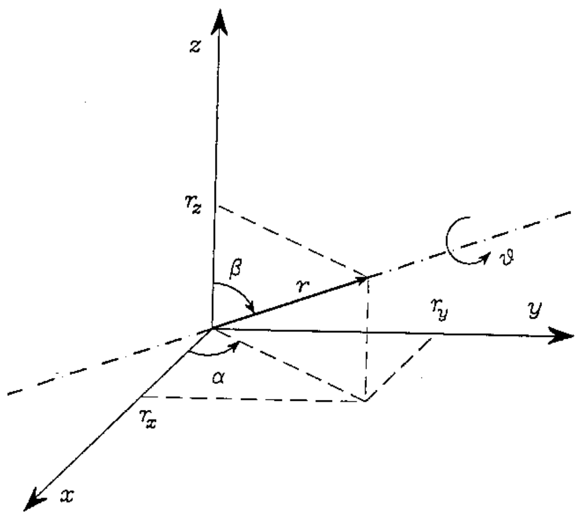

FIGURE 2.8
Rotation about an arbitrary axis.

axis is[2]

$$\boldsymbol{R}_r(\vartheta) = \begin{bmatrix} r_x^2(1-c_\vartheta)+c_\vartheta & r_x r_y(1-c_\vartheta)-r_z s_\vartheta & r_x r_z(1-c_\vartheta)+r_y s_\vartheta \\ r_x r_y(1-c_\vartheta)+r_z s_\vartheta & r_y^2(1-c_\vartheta)+c_\vartheta & r_y r_z(1-c_\vartheta)-r_x s_\vartheta \\ r_x r_z(1-c_\vartheta)-r_y s_\vartheta & r_y r_z(1-c_\vartheta)+r_x s_\vartheta & r_z^2(1-c_\vartheta)+c_\vartheta \end{bmatrix}. \tag{2.19}$$

Hence, for given $\boldsymbol{r}$ and ϑ, Eq. (2.19) represents the corresponding rotation matrix. For this matrix, the following property holds:

$$\boldsymbol{R}_{-r}(-\vartheta) = \boldsymbol{R}_r(\vartheta), \tag{2.20}$$

i.e., such representation is not unique, since a rotation by $-\vartheta$ about $-\boldsymbol{r}$ produces the same effects as those of a rotation by ϑ about $\boldsymbol{r}$.

If it is desired to solve the *inverse problem* to compute the axis and angle associated with a given rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

[2] The notations c_ϑ and s_ϑ are the abbreviations for $\cos\vartheta$ and $\sin\vartheta$, respectively; short-hand notations of this kind will be adopted often throughout the text.

the following result is useful:

$$\vartheta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$
$$\boldsymbol{r} = \frac{1}{2\sin\vartheta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}, \tag{2.21}$$

for $\sin\vartheta \neq 0$. Notice that Eqs. (2.21) express the rotation in terms of four parameters; namely, the angle and the three components of the axis unit vector. However, it can be observed that the three components of $\boldsymbol{r}$ are not independent but are constrained by the condition

$$r_x^2 + r_y^2 + r_z^2 = 1. \tag{2.22}$$

If $\sin\vartheta = 0$, Eqs. (2.21) become meaningless. To solve the inverse problem, it is necessary to directly refer to the particular expressions attained by the rotation matrix $\boldsymbol{R}$ and find the solving formulæ in the two cases $\vartheta = 0$ and $\vartheta = \pi$. Notice that, when $\vartheta = 0$ (null rotation), the unit vector $\boldsymbol{r}$ is arbitrary.

2.5 MINIMAL REPRESENTATIONS OF ORIENTATION

Rotation matrices in general give a redundant description of frame orientation; in fact, they are characterized by nine elements which are not independent but related by six constraints due to the orthogonality conditions given in (2.4). Even in the case of describing orientation in terms of rotation about an arbitrary axis, a representation in terms of four parameters is obtained: the angle and the three components of the axis unit vector. These components are not independent but are constrained by the unit norm condition given in (2.22).

This implies that the actual free parameters to describe orientation are *three*. A representation of orientation in terms of three independent parameters constitutes a *minimal representation*. The problem of determining a minimal representation admits different solutions. In the following, the rotation matrices obtained with the representations in terms of *Euler angles* and *RPY angles* are analyzed.

2.5.1 Euler Angles

The *Euler angles* constitute a minimal representation of orientation obtained by composing elementary rotations expressed with respect to axes of current frames. There exist twelve distinct sets of Euler angles, with regard to the sequence of possible elementary rotations; below, the so-called ZYZ representation is considered.

Let $(\varphi, \vartheta, \psi)$ be the given set of Euler angles. The overall rotation described by these angles is obtained as composition of the following elementary rotations (Fig. 2.9):

- Rotate the reference frame by the angle φ about axis z; this rotation is described by the rotation matrix $\boldsymbol{R}_z(\varphi)$ which is formally defined in (2.6).

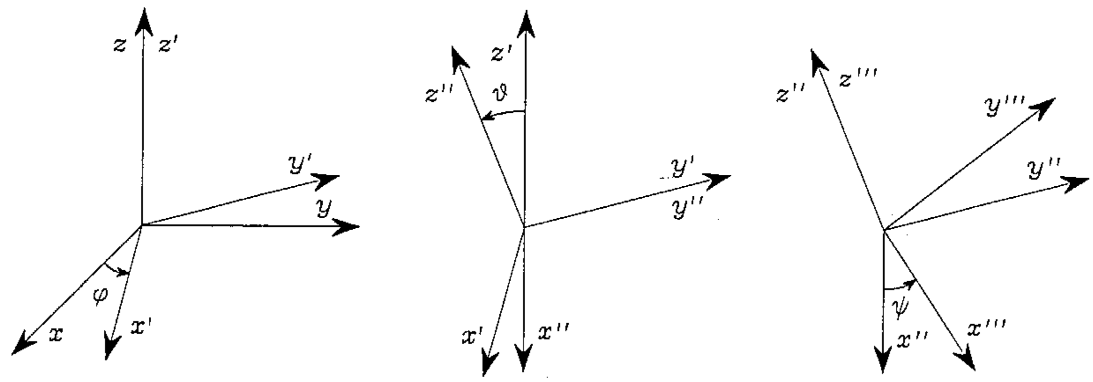

FIGURE 2.9
Representation of Euler angles ZYZ.

- Rotate the current frame by the angle ϑ about axis y'; this rotation is described by the rotation matrix $\boldsymbol{R}_{y'}(\vartheta)$ which is formally defined in (2.7).
- Rotate the current frame by the angle ψ about axis z''; this rotation is described by the rotation matrix $\boldsymbol{R}_{z''}(\psi)$ which is again formally defined in (2.6).

The resulting frame orientation is obtained by composition of rotations with respect to *current frames*, and then it can be computed via postmultiplication of the matrices of elementary rotation, i.e.,

$$\begin{aligned}\boldsymbol{R}_{\text{EUL}} &= \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)\\ &= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}.\end{aligned} \tag{2.23}$$

It is useful to solve the *inverse problem*, that is to determine the set of Euler angles corresponding to a given rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

Compare this expression with that of $\boldsymbol{R}_{\text{EUL}}$ in (2.23). By considering the elements $[1,3]$ and $[2,3]$, on the assumption that $r_{13} \neq 0$ and $r_{23} \neq 0$, it follows that

$$\varphi = \text{Atan2}(r_{23}, r_{13}),$$

where $\text{Atan2}(y, x)$ is the arctangent function of two arguments[3]. Then, squaring and

[3] The function $\text{Atan2}(y, x)$ computes the arctangent of the ratio y/x but utilizes the sign of each argument to determine which quadrant the resulting angle belongs to; this allows the correct determination of an angle in a range of 2π.