

Name: _____

Grade: _____

Final Exam

CPSC 411, Spring 2009

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- This exam contains 8 problems. You have 80 minutes to earn up to 100 points.
- This exam is closed book.
- You are allowed to use a nonprogrammable calculator, although you will not need one.
- You are not allowed to use cell-phones, iphones, and other wireless devices.
- Do not spend too much time on a single problem. Read them all through first and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.
- Recall the Aggie code of honor. Cheating will have severe consequences.

The work shown in this exam is my own.

Signature required: _____

Problem 1 (2 Points)

Write (print) your name on each odd-numbered page.

Problem 2 (18 Points)

Suppose you have n matrices M_1, \dots, M_n , where the matrix M_k is of dimension $p_{k-1} \times p_k$.

(a) Using the standard matrix multiplication algorithm, how many scalar multiplications are needed to multiply matrices M_k and M_{k+1} ?

Let $m(i, j)$ denote minimal number of scalar multiplications needed to form the matrix product $M_i M_{i+1} \cdots M_j$ using standard matrix multiplication and an optimal way to set parentheses. One can express $m(i, j)$ in terms of $m(i, k)$ and $m(k + 1, j)$ for some k in the range $i \leq k < j$.

(b) Find the recurrence relation for $m(i, j)$ using $m(i, k)$ and $m(k + 1, j)$.

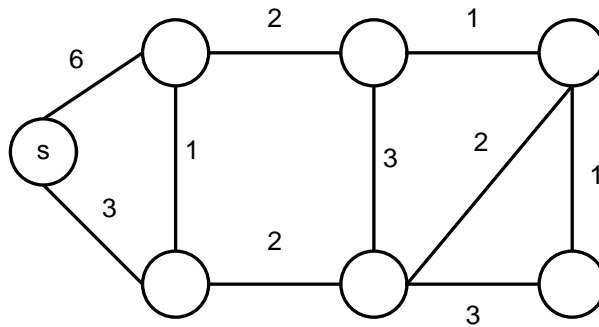
$$m(i, j) = \begin{cases} & \text{if } i = j \\ & \text{if } i < j. \end{cases}$$

(c) What is the time complexity (in Θ -notation) of the dynamic programming solution based on the recurrence given in part (b) to find the optimal way to multiply the n matrices M_1, \dots, M_n ?

(d) In what order do you need to compute the entries $m(i, j)$ to determine $m(1, n)$? [Illustrate your answer.]

Problem 3 (20 Points)

(a) Use Kruskal's algorithm to determine the minimum spanning tree of the following graph. Use the node labeled s in case the algorithm needs a starting node.



(b) Is the minimum spanning tree in part (a) uniquely determined? Explain.

(c) What is the time complexity of Kruskal's algorithm on input of a graph $G = (V, E)$?

(d) Does Kruskal's algorithm require that the graph is connected? Explain.

Problem 4 (15 Points)

Consider the following two decision problems.

P1 Input: A set of n squares, specified by their corner points.

Question: Is there any point in the plane that is covered by k or more squares?

P2 Input: A graph $G = (V, E)$ with n vertices, and a number k .

Question: Is there a set of k mutually adjacent vertices?

Clearly, both problems are in NP. There exists a simple translation from **P1** to **P2**: Make a vertex for each square and add an edge between a pair of vertices if the corresponding squares overlap.

(a) If **P1** is NP-complete, would this translation imply that **P2** is NP-complete? Give a short and clear explanation.

(b) If **P2** is NP-complete, would this translation imply that **P1** is NP-complete? Give a short and clear explanation.

(c) If we replace squares by circles in **P1**, is then the simple translation still valid? For instance, does the existence of a triangle in the graph mean that there exists a point covered by at least three circles?

Problem 5 (10 Points)

Recall that a property about programs is called *functional* if it just refers to the language accepted by the program and not about the specific code of the program. A functional property about programs is *nontrivial* if some programs have the property and some do not. Rice's theorem asserts that nontrivial functional properties of programs are undecidable.

Consider the property **O**: "The program accepts the input 0".

(a) Is the property **O** functional?

(b) Is the property **O** nontrivial?

(c) Is the property **O** decidable?

Problem 6 (15 Points)

(a) Determine the greatest common divisor $\gcd(101, 17)$ using the Euclidean algorithm. [Use the format $\gcd(a, b) = \gcd(b, c) = \dots$]

(b) Let x and n be given integers such that $\gcd(x, n) = 1$. Prove that there exists an integer y such that $xy \equiv 1 \pmod{n}$.

(c) Keeping the assumptions of part (b), explain how one can efficiently find such an integer y satisfying $xy \equiv 1 \pmod{n}$.

Problem 7 (10 Points)

A customer is willing to accept a randomized algorithm solving a given problem that has an expected running time of at most one hour. Your randomized algorithm solves the problem with 10% probability of success in about 5 minutes running time. If you repeat until you have success, does this solution meet the requirement of your customer? [Formalize this problem using random variables and prove your answer]

Problem 8 (10 Points)

Recall that a matroid (S, \mathcal{F}) is a (i) nonempty finite set S and a family \mathcal{F} of subsets of S such that (ii) $B \in \mathcal{F}$ and $A \subset B$ implies $A \in \mathcal{F}$, and (iii) if $A, B \in \mathcal{F}$, and $|A| < |B|$, then there exists some element $x \in B$ such that $A \cup \{x\} \in \mathcal{F}$.

Prove that if (S, \mathcal{F}) is a matroid, then (S, \mathcal{F}') , with

$$\mathcal{F}' = \{A' : S \setminus A' \text{ contains some maximal } A \in \mathcal{F}\},$$

is a matroid as well.