Calculus of Finite Differences

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Motivation

When we analyze the runtime of algorithms, we simply count the number of operations. For example, the following loop

for k = 1 to n do

    square(k);

where $\text{square}(k)$ is a function that has running time $T_2 k^2$. Then the total number of instructions is given by

$$T_1(n + 1) + \sum_{k=1}^{n} T_2 k^2$$

where $T_1$ is the time for loop increment and comparison.
Motivation

The question is how to find closed form representations of sums such as

\[
\sum_{k=1}^{n} k^2
\]

Of course, you can look up this particular sum. Perhaps you can even guess the solution and prove it by induction. However, neither of these “methods” are entirely satisfactory.
Motivation

The sum

$$\sum_{k=a}^{b} g(k)$$

may be regarded as a discrete analogue of the integral

$$\int_{a}^{b} g(x) dx$$

We can evaluate the integral by finding a function $f(x)$ such that $\frac{d}{dx} f(x) = g(x)$, since the fundamental theorem of calculus yields

$$\int_{a}^{b} g(x) dx = f(b) - f(a).$$
Motivation

We would like to find a result that is analogous to the fundamental theorem of calculus for sums. The calculus of finite differences will allow us to find such a result.

Some benefits:

- Closed form evaluation of certain sums.

- The calculus of finite differences will explain the real meaning of the Harmonic numbers (and why they occur so often in the analysis of algorithms).
**Difference Operator**

Given a function $g(n)$, we define the difference operator $\Delta$ as

$$\Delta g(n) = g(n + 1) - g(n)$$

Let $E$ denote the shift operator $Eg(n) = g(n + 1)$, and $I$ the identity operator. Then

$$\Delta = E - I$$
Examples

a) Let \( f(n) = n \). Then

\[ \Delta f(n) = n + 1 - n = 1. \]

b) Let \( f(n) = n^2 \). Then

\[ \Delta f(n) = (n + 1)^2 - n^2 = 2n + 1. \]

c) Let \( f(n) = n^3 \). Then

\[ \Delta f(n) = (n + 1)^3 - n^3 = 3n^2 + 3n + 1. \]
Falling Power

We define the \( m \)-th falling power of \( n \) as

\[
n^m = n(n-1) \cdots (n-m+1)
\]

for \( m \geq 0 \). We have

\[
\Delta n^m = mn^{m-1}.
\]
Falling Power

**Theorem.** We have

\[ \Delta n^m = mn^{m-1}. \]

**Proof.** By definition,

\[ \Delta n^m = (n + 1)n \cdots (n - m + 2) \]
\[ -n \cdots (n - m + 2)(n - m + 1) \]
\[ = mn \cdots (n - m + 2) \]
Negative Falling Powers

Since

\[ \frac{n^m}{n^{m-1}} = (n - m + 1), \]

we have

\[ \frac{n^2}{n^1} = n(n - 1)/n = (n - 1), \]

\[ \frac{n^1}{n^0} = n/1 = n \]

so we expect that

\[ \frac{n^0}{n^{-1}} = n + 1 \]

holds, which implies that

\[ n^{-1} = 1/(n + 1). \]
Similarly, we want

\[ \frac{n^{-1}}{n^{-2}} = n + 2 \]

so

\[ n^{-2} = \frac{1}{(n + 1)(n + 2)} \]

We define

\[ n^{-m} = \frac{1}{(n + 1)(n + 2) \cdots (n + m)} \]
Exercise

Show that for $m \geq 0$, we have

$$\Delta n^{-m} = -mn^{-m-1}$$
Let $c \neq 1$ be a fixed real number. Then

$$\Delta c^n = c^{n+1} - c^n = (c - 1)c^n.$$ 

In particular,

$$\Delta 2^n = 2^n.$$
Antidifference

A function \( f(n) \) with the property that

\[ \Delta f(n) = g(n) \]

is called the **antidifference** of the function \( g(n) \).

**Example.** The antidifference of the function \( g(n) = n^m \) is given by

\[ f(n) = \frac{1}{m + 1} n^{m+1}. \]
Example. The antidifference of the function $g(n) = c^n$ is given by

$$f(n) = \frac{1}{c - 1} c^n.$$ 

Indeed,

$$\Delta f(n) = \frac{1}{c - 1} (c^{n+1} - c^n) = c^n.$$
Theorem. Let \( f(n) \) be an antiderivative of \( g(n) \). Then
\[
\sum_{n=a}^{b} g(n) = f(b + 1) - f(a).
\]

Proof. We have
\[
\sum_{n=a}^{b} g(n) = \sum_{n=a}^{b} \Delta f(n)
\]
\[
= \sum_{n=a}^{b} (f(n + 1) - f(n))
\]
\[
= \sum_{n=a+1}^{b+1} f(n) - \sum_{n=a}^{b} f(n)
= f(b + 1) - f(a).
\]
Example 1

Suppose we want to find a closed form for the sum

$$\sum_{n=5}^{64} c^n.$$ 

An antiderivative of $c^n$ is $\frac{1}{c-1}c^n$. Therefore, by the fundamental theorem of finite difference, we have

$$\sum_{n=5}^{64} c^n = \frac{1}{c-1}c^n \bigg|_{5}^{65} = \frac{c^{65} - c^5}{c - 1}.$$
Antidifference

We are going to denote an antidifference of a function $f(n)$ by

$$\sum f(n) \delta n.$$  

The $\delta n$ plays the same role as the $dx$ term in integration.

For example,

$$\sum n^m \delta n = \frac{1}{m + 1} n^{m+1}$$

when $m \neq -1$. What about $m = -1$?
Harmonic Numbers = Discrete ln

We have

$$\sum n^{-1} \delta n = H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$ 

Indeed,

$$\Delta H_n = H_{n+1} - H_n = \frac{1}{n+1} = n^{-1}.$$ 

Thus, the antidifference of \( n^{-1} \) is \( H_n \).
Let \( f(n) \) and \( g(n) \) be two sequences and \( a \) and \( b \) two constants. Then

\[
\Delta(af(n) + bg(n)) = a \Delta f(n) + b \Delta g(n).
\]

Consequently, the antidifferences are linear as well:

\[
\sum (af(n) + bg(n)) \delta n = a \sum f(n) \delta n + b \sum g(n) \delta n
\]
Example

To solve our motivating example, we need to find a closed form for the sum

\[ \sum_{k=1}^{n} k^2. \]

Since \( k^2 = k^2 + k^1 \), an antiderivative of \( k^2 \) is given by

\[ \sum k^2 \delta k = \sum (k^2 + k^1) \delta k = \frac{1}{3} k^3 + \frac{1}{2} k^2. \]

Thus, the sum

\[ \sum_{k=1}^{n} k^2 = \frac{1}{3} k^{3+1} + \frac{1}{2} k^{2+1} = \cdots = \frac{n(2n + 1)(n + 1)}{6}. \]
Binomial Coefficients

By Pascal’s rule for binomial coefficients, we have

\[
\binom{n}{k} + \binom{n}{k + 1} = \binom{n + 1}{k + 1}.
\]

Therefore,

\[
\Delta \binom{n}{k + 1} = \binom{n}{k}.
\]

In other words,

\[
\sum \binom{n}{k} \delta n = \binom{n}{k + 1}.
\]

For example, this shows that

\[
\sum_{n=0}^{m} \binom{n}{k} = \binom{m + 1}{k + 1} - \binom{0}{k + 1} = \binom{m + 1}{k + 1}.
\]