Undecidability

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[based on slides by Prof. Welch]
Sources

Understanding Limits of Computing

- So far, we have studied how efficiently various problems can be solved.
- There has been no question as to whether it is possible to solve the problem.
- If we want to explore the boundary between what can and what cannot be computed, we need a model of computation.
Models of Computation

- Need a way to clearly and unambiguously specify how computation takes place
- Many different mathematical models have been proposed:
  - Turing Machines
  - Random Access Machines
  - ...
- They have all been found to be equivalent!
Church-Turing Thesis

• Conjecture: Anything we reasonably think of as an algorithm can be computed by a Turing Machine (specific formal model).

• So we might as well think in our favorite programming language, or in pseudocode.
• Frees us from the tedium of having to provide boring details
  • in principle, pseudocode descriptions can be converted into some appropriate formal model
Short Review of some Basic Set Theory Concepts
Some Notation

If A and B are sets, then the set of all functions from A to B is denoted by $B^A$.

If A is a set, then $P(A)$ denotes the power set, i.e., $P(A)$ is the set of all subsets of A.
Cardinality

Two sets $A$ and $B$ are said to have the same cardinality if and only if there exists a bijective function from $A$ onto $B$.

[ A function is bijective if it is one-to-one and onto ]

We write $|A|=|B|$ if $A$ and $B$ have the same cardinality.

[Note that $|A|=|B|$ says that $A$ and $B$ have the same number of elements, even if we do not yet know about numbers!]
How Set Theorists Count

Set theorists count

- $0 = \{\} \quad // \text{the empty set exists by axiom}$
  
  This set contains no elements

- $1 = \{0\} = \{\{\}\} \quad // \text{form the set containing } \{\}$
  
  This set contains one element

- $2 = \{0, 1\} = \{\{\}, \{\{\}\}\} \quad // \text{form the set containing } \{\{\}, \{\{\}\}\}$
  
  This set contains two elements

- Keep including all previously created sets as elements of the next set.
Example

Theorem: $|P(X)| = |2^X|$ 

Proof: The bijection from $P(X)$ onto $2^X$ is given by the characteristic function. q.e.d.

Example: $X = \{a,b\}$

$\emptyset$ corresponds to $f(a)=0, f(b)=0$

$\{a\}$ corresponds to $f(a)=1, f(b)=0$

$\{b\}$ corresponds to $f(a)=0, f(b)=1$

$\{a,b\}$ corresponds to $f(a)=1, f(b)=1$
More About Cardinality

Let $A$ and $B$ be sets.

We write $|A| \leq |B|$ if and only if there exists an injective function from $A$ to $B$.

We write $|A| < |B|$ if and only if there exist an injective function from $A$ to $B$, but no bijection exists from $A$ to $B$. 
Cardinality

Cantor’s Theorem: Let $S$ be any set. Then $|S| < |P(S)|$.

Proof: Since the function $i$ from $S$ to $P(S)$ given by $i(s) = \{s\}$ is injective, we have $|S| \leq |P(S)|$.

Claim: There does not exist any function $f$ from $S$ to $P(S)$ that is surjective.

Indeed, $T = \{ s \in S : s \notin f(s) \}$ is not contained in $f(S)$. An element $s$ in $S$ is either contained in $T$ or not.

- If $s \in T$, then $s \notin f(s)$ by definition of $T$. Thus, $T \neq f(s)$.
- If $s \notin T$, then $s \in f(s)$ by definition of $T$. Thus, $T \neq f(s)$.

Therefore, $f$ is not surjective. This proves the claim.
Uncountable Sets and Uncomputable Functions
Countable Sets

Let $N$ be the set of natural numbers.

A set $X$ is called **countable** if and only if there exists a surjective function from $N$ onto $X$.

Thus, finite sets are countable, $N$ is countable, but the set of real numbers is not countable.
An Uncountable Set

Theorem: The set $\mathbb{N}^\mathbb{N} = \{ f \mid f: \mathbb{N} \to \mathbb{N} \}$ is not countable.

Proof: We have $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$ by Cantor’s theorem. Since $|\mathcal{P}(\mathbb{N})| = |2^\mathbb{N}|$ and $2^\mathbb{N}$ is a subset of $\mathbb{N}^\mathbb{N}$ we can conclude that

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |2^\mathbb{N}| \leq |\mathbb{N}^\mathbb{N}|.$$ q.e.d.
Alternate Proof:
The Set $\mathbb{N}^\mathbb{N}$ is Uncountable

Seeking a contradiction, we assume that the set of functions from $\mathbb{N}$ to $\mathbb{N}$ is countable.
Let the functions in the set be $f_0, f_1, f_2, \ldots$
We will obtain our contradiction by defining a function $f^d$ (using "diagonalization") that should be in the set but is not equal to any of the $f_i$'s.
Diagonalization

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Diagonalization

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• In the example:
Diagonalization

- Define the function:  \( f^d(n) = f_n(n) + 1 \)
- In the example:
  - \( f^d(0) = 4 + 1 = 5 \), so \( f^d \neq f_0 \)
Diagonalization

- Define the function: \( f^d(n) = f_n(n) + 1 \)
- In the example:
  - \( f^d(0) = 4 + 1 = 5 \), so \( f^d \neq f_0 \)
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Diagonalization

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  - \( f^d(2) = 5 + 1 = 6 \), so \( f^d \neq f_2 \)
Diagonalization

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  - \( f^d(3) = 7 + 1 = 8 \), so \( f^d \neq f_3 \)
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  - \( f^d(3) = 7 + 1 = 8 \), so \( f^d \neq f_3 \)
  - \( f^d(4) = 3 + 1 = 4 \), so \( f^d \neq f_4 \)
Diagonalization

- Define the function: $f^d(n) = f_n(n) + 1$
- In the example:
  - $f^d(0) = 4 + 1 = 5$, so $f^d \neq f_0$
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  - $f^d(2) = 5 + 1 = 6$, so $f^d \neq f_2$
  - $f^d(3) = 7 + 1 = 8$, so $f^d \neq f_3$
  - $f^d(4) = 3 + 1 = 4$, so $f^d \neq f_4$
  - etc.
Uncomputable Functions Exist!

Consider all programs (in our favorite model) that compute functions in $\mathbb{N}^\mathbb{N}$.

The set $\mathbb{N}^\mathbb{N}$ is uncountable, hence cannot be enumerated.

However, the set of all programs can be enumerated (i.e., is countable).

Thus there must exist some functions in $\mathbb{N}^\mathbb{N}$ that cannot be computed by a program.
Set of All Programs is Countable

- Fix your computational model (e.g., programming language).
- Every program is finite in length.
- For every integer \( n \), there is a finite number of programs of length \( n \).
- Enumerate programs of length 1, then programs of length 2, then programs of length 3, etc.
Uncomputable Functions

- Previous proof just showed there must exist uncomputable functions
- Did not exhibit any particular uncomputable function
- Maybe the functions that are uncomputable are uninteresting...
- But actually there are some VERY interesting functions (problems) that are uncomputable
The Halting Problem
The Function Halt
The Function Halt

- Consider this function, called Halt:
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  - input: code for a program P and an input X for P
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- Consider this function, called Halt:
  - input: code for a program P and an input X for P
  - output: 1 if P terminates (halts) when executed on input X, and 0 if P doesn't terminate (goes into an infinite loop) when executed on input X
The Function Halt

- Consider this function, called Halt:
  - input: code for a program P and an input X for P
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- By the way, a compiler is a program that takes as input the code for another program
The Function Halt

Consider this function, called Halt:

- input: code for a program $P$ and an input $X$ for $P$
- output: 1 if $P$ terminates (halts) when executed on input $X$, and 0 if $P$ doesn't terminate (goes into an infinite loop) when executed on input $X$

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Note that the input $X$ to $P$ could be (the code for) $P$ itself.
The Function Halt

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In the compiler example, a compiler can be run on its own code.
The Function Halt

- We can view Halt as a function from $\mathbb{N}$ to $\mathbb{N}$:
  - $P$ and $X$ can be represented in ASCII, which is a string of bits.
  - This string of bits can also be interpreted as a natural number.
- The function Halt would be a useful diagnostic tool in debugging programs
Halt is Uncomputable

• Suppose in contradiction there is a program $P_{\text{halt}}$ that computes Halt.

• Use $P_{\text{halt}}$ as a subroutine in another program, $P_{\text{self}}$.

• Description of $P_{\text{self}}$:
  • input: code for any program $P$
  • constructs pair $(P,P)$ and calls $P_{\text{halt}}$ on $(P,P)$
  • returns same answer as $P_{\text{halt}}$
$P_{\text{self}}$

- $P_{\text{self}}$: The input $P$.
- $(P, P)$: The pair $(P, P)$.
- $P_{\text{halt}}$: The output $P_{\text{halt}}$.

0 if $P$ halts on input $P$

1 if $P$ doesn't halt on input $P$
Halt is Uncomputable

• Now use $P_{\text{self}}$ as a subroutine inside another program $P_{\text{diag}}$.

• Description of $P_{\text{diag}}$:
  • input: code for any program $P$
  • call $P_{\text{self}}$ on input $P$
  • if $P_{\text{self}}$ returns 1 then go into an infinite loop
  • if $P_{\text{self}}$ returns 0 then output 0

• $P_{\text{diag}}$ on input $P$ does the opposite of what program $P$ does on input $P$
A Turing machine $P_{\text{diag}}$ is shown in the diagram. This machine takes two inputs: $P$ and another instance of the machine $P$. It outputs 1 if $P$ halts on input $P$, and 0 if it doesn't halt on input $P$. The machine is self-referential, as indicated by the $(P,P)$ input leading to $P_{\text{self}}$.
Halt is Uncomputable
Halt is Uncomputable

- Review behavior of $P_{\text{diag}}$ on input $P$: 
Halt is Uncomputable

- Review behavior of $P_{\text{diag}}$ on input $P$:
  - If $P$ halts when executed on input $P$, then $P_{\text{diag}}$ goes into an infinite loop
Halt is Uncomputable

- Review behavior of \( P_{\text{diag}} \) on input \( P \):
  - If \( P \) halts when executed on input \( P \), then \( P_{\text{diag}} \) goes into an infinite loop
  - If \( P \) does not halt when executed on input \( P \), then \( P_{\text{diag}} \) halts (and outputs 0)
Halt is Uncomputable

• Review behavior of $P_{\text{diag}}$ on input $P$:
  • If $P$ halts when executed on input $P$, then $P_{\text{diag}}$ goes into an infinite loop
  • If $P$ does not halt when executed on input $P$, then $P_{\text{diag}}$ halts (and outputs 0)

• What happens if $P_{\text{diag}}$ is given its own code as input? It either halts or doesn't.
Halt is Uncomputable

- Review behavior of $P_{diag}$ on input $P$:
  - If $P$ halts when executed on input $P$, then $P_{diag}$ goes into an infinite loop
  - If $P$ does not halt when executed on input $P$, then $P_{diag}$ halts (and outputs 0)

- What happens if $P_{diag}$ is given its own code as input? It either halts or doesn't.
  - If $P_{diag}$ halts when executed on input $P_{diag}$, then $P_{diag}$ goes into an infinite loop
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Contradiction
Halt is Uncomputable

• What went wrong?

• Our assumption that there is an algorithm to compute Halt was incorrect.

• So there is no algorithm that can correctly determine if an arbitrary program halts on an arbitrary input.
Undecidability
Undecidability

• The analog of an uncomputable function is an **undecidable set**.

• The theory of what can and cannot be computed focuses on identifying sets of strings:
  
  • an algorithm is required to "decide" if a given input string is in the set of interest
  
  • similar to deciding if the input to some NP-complete problem is a YES or NO instance
Undecidability

• Recall that a (formal) language is a set of strings, assuming some encoding.

• Analogous to the function Halt is the set $H$ of all strings that encode a program $P$ and an input $X$ such that $P$ halts when executed on $X$.

• There is no algorithm that can correctly identify for every string whether it belongs to $H$ or not.
More Reductions

For NP-completeness, we were concerned with (time) complexity of problems:
- reduction from P1 to P2 had to be fast (polynomial time)

Now we are concerned with computability of problems:
- reduction from P1 to P2 just needs to be computable, don't care how slow it is
Many-One Reduction

all strings over $L_1$'s alphabet

$L_1$

$\mathcal{f}$

all strings over $L_2$'s alphabet

$L_2$
Many-One Reduction

all strings over $L_1$'s alphabet

all strings over $L_2$'s alphabet

$f$
Many-One Reduction

all strings over $L_1$'s alphabet

all strings over $L_2$'s alphabet

$f$
Many-One Reduction

all strings over $L_1$'s alphabet

$L_1$

all strings over $L_2$'s alphabet

$L_2$

$f$
Many-One Reduction

all strings over $L_1$'s alphabet

$\Rightarrow$

all strings over $L_2$'s alphabet

$f$
Many-One Reduction

all strings over $L_1$'s alphabet

all strings over $L_2$'s alphabet
Many-One Reduction

all strings over $L_1$'s alphabet

$L_1$

$f$

all strings over $L_2$'s alphabet

$L_2$
Many-One Reduction

all strings over $L_1$'s alphabet

$\mathbb{L}_1$

$\mathbb{L}_2$

all strings over $L_2$'s alphabet

$f$
Many-One Reduction

- YES instances map to YES instances
- NO instances map to NO instances
- computable (doesn't matter how slow)
- Notation: $L_1 \leq_m L_2$
- Think: $L_2$ is at least as hard to compute as $L_1$
Many-One Reduction Theorem

**Theorem:** If $L_1 \leq_m L_2$ and $L_2$ is computable, then $L_1$ is computable.

**Proof:** Let $f$ be the many-one reduction from $L_1$ to $L_2$. Let $A_2$ be an algorithm for $L_2$. Here is an algorithm $A_1$ for $L_1$.

- **input:** $x$
- **compute** $f(x)$
- **run** $A_2$ on input $f(x)$
Implication

- If there is no algorithm for $L_1$, then there is no algorithm for $L_2$.
- In other words, if $L_1$ is undecidable, then $L_2$ is also undecidable.
- Pay attention to the direction!
Example of a Reduction

- Consider the language $L_{\text{NE}}$ consisting of all strings that encode a program that halts (does not go into an infinite loop) on at least one input.
- Use a reduction to show that $L_{\text{NE}}$ is not decidable:
  - Show some known undecidable language $\leq_m L_{\text{NE}}$.
  - Our only choice for the known undecidable language is $H$ (the language corresponding to the halting problem).
  - So show $H \leq_m L_{\text{NE}}$. 
Example of a Reduction

- Given an arbitrary H input (encoding of a program $P$ and an input $X$ for $P$), compute an $L_{NE}$ input (encoding of a program $P'$)
  - such that $P$ halts on input $X$ if and only if $P'$ halts on at least one input.
- Construction consists of **writing code** to describe $P'$.
- What should $P'$ do? It's allowed to use $P$ and $X$
Example of a Reduction

- The code for $P'$ does this:
  - input $X'$:
  - ignore $X'$
  - call program $P$ on input $X$
  - if $P$ halts on input $X$ then return whatever $P$ returns

- How does $P'$ behave?
  - If $P$ halts on $X$, then $P'$ halts on every input
  - If $P$ does not halt on $X$, then $P'$ does not halt on any input
Example of a Reduction

- Thus if \((P,X)\) is a YES input for \(H\) (meaning \(P\) halts on input \(X\)), then \(P'\) is a YES input for \(L_{\text{NE}}\) (meaning \(P'\) halts on at least one input).

- Similarly, if \((P,X)\) is NO input for \(H\) (meaning \(P\) does not halt on input \(X\)), then \(P'\) is a NO input for \(L_{\text{NE}}\) (meaning \(P'\) does not halt on even one input).

- Since \(H\) is undecidable, and we showed \(H \leq_m L_{\text{NE}}\), \(L_{\text{NE}}\) is also undecidable.
Generalizing Such Reductions

• There is a way to generalize the reduction we just did, to show that lots of other languages that describe properties of programs are also undecidable.

• Focus just on programs that accept languages (sets of strings):
  • I.e., programs that say YES or NO about their inputs
  • Ex: a compiler tells you YES or NO whether its input is syntactically correct
Properties About Programs

- Define a *property about programs* to be a set of strings that encode some programs.
  - The "property" corresponds to whatever it is that all the programs have in common

Example:
- Program terminates in 10 steps on input y
- Program never goes into an infinite loop
- Program accepts a finite number of strings
- Program contains 15 variables
- Program accepts 0 or more inputs
Functional Properties

• A property about programs is called **functional** if it just refers to the language accepted by the program and not about the specific code of the program
  • Program terminates in 10 steps on input y (n.f.)
  • Program never goes into an infinite loop (f.)
  • Program accepts a finite number of strings (f.)
  • Program contains 15 variables (n.f.)
Nontrivial Properties

- A functional property about programs is **nontrivial** if some programs have the property and some do not.
- Example of nontrivial programs:
  - Program never goes into an infinite loop
  - Program accepts a finite number of strings
- Example of a trivial program:
  - Program accepts 0 or more inputs
**Rice's Theorem**

- Every nontrivial (functional) property about programs is undecidable.
- The proof is a generalization of the reduction shown earlier.
- Very powerful and useful theorem:
  - To show that some property is undecidable, only need to show that is nontrivial and functional, then appeal to Rice's Theorem
Applying Rice's Theorem

- Consider the property "program accepts a finite number of strings".
- This property is functional:
  - it is about the language accepted by the program and not the details of the code of the program
- This property is nontrivial:
  - Some programs accept a finite number of strings (for instance, the program that accepts no input)
  - Some accept an infinite number (for instance, the program that accepts every input)
- By Rice's theorem, the property is undecidable.
Implications of Undecidable Program Property

- It is not possible to design an algorithm (write a program) that can analyze any input program and decide whether the input program satisfies the property!
- Essentially all you can do is simulate the input program and see how it behaves
  - but this leaves you vulnerable to an infinite loop