3SAT

Andreas Klappenecker
[partially based on slides by Jennifer Welch]
Given a boolean function in conjunctive normal form such that every clause contains exactly three literals, decide whether the formula is satisfiable.

[This a special case of SAT]
Proving NP-Completeness

How do you prove that a decision problem $L$ is NP-complete?

1. Show that $L$ is in NP.
2. (a) Choose an appropriate known NP-complete language $L'$.
   (b) Show $L' \leq_p L$
Proof Strategy

(1) 3SAT is in NP, since we can check in polynomial time whether a given truth assignment evaluates to true.

(2.a) Choose SAT as a known NP-complete problem.

(2.b) Describe a reduction from SAT inputs to 3SAT inputs

- computable in polynomial time
- SAT input is satisfiable iff constructed 3SAT input is satisfiable
General Idea of the Reduction

We're given an arbitrary CNF formula $C = c_1 \land c_2 \land \ldots \land c_m$ over set of variables, where each $c_i$ is a clause (a disjunction of literals).

We will replace each clause $c_i$ with a conjunction of clauses $c_i'$, and may use some extra variables. Each clause in $c_i'$ will have exactly 3 literals. The transformed input will be conjunction of all the clauses in all the $c_i'$. 
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 1: $k = 1$. Use extra variables $y_i^1$ and $y_i^2$. Replace $c_i$ with 4 clauses:

$$(z_1 \lor y_i^1 \lor y_i^2) \land (z_1 \lor \neg y_i^1 \lor y_i^2) \land (z_1 \lor y_i^1 \lor \neg y_i^2) \land (z_1 \lor \neg y_i^1 \lor \neg y_i^2).$$
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor ... \lor z_k$

Case 2: $k = 2$. Use extra variable $y_i\bar{1}$. Replace $c_i$ with 2 clauses:

$$(z_1 \lor z_2 \lor \neg y_i\bar{1}) \land (z_1 \lor z_2 \lor y_i\bar{1}).$$
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor ... \lor z_k$

Case 3: $k = 3$. No extra variables are needed.

Keep $c_i$: $(z_1 \lor z_2 \lor z_3)$
Reduction from SAT to 3SAT

Let \( c_i = z_1 \lor z_2 \lor ... \lor z_k \)

Case 4: \( k > 3 \). Use extra variables \( y_{i1}, ..., y_{i(k-3)} \). Replace \( c_i \) with \( k-2 \) clauses:

\[
(z_1 \lor z_2 \lor y_{i1}) \land (-y_{i1} \lor z_3 \lor y_{i2}) \land (-y_{i2} \lor z_4 \lor y_{i3}) \land ... \\
\land (-y_{i(k-5)} \lor z_{k-3} \lor y_{i(k-4)}) \land (-y_{i(k-4)} \lor z_{k-2} \lor y_{i(k-3)}) \\
\land (-y_{i(k-3)} \lor z_{k-1} \lor z_k)
\]
Polynomial Time Reduction

Each new formula is at most a constant times larger than the original formula, and the translation is straightforward. Therefore, the reduction is polynomial time.
Correctness of the Reduction

Show that CNF formula C is satisfiable iff the 3-CNF formula C' constructed is satisfiable.

=>: Suppose that C is satisfiable. We need to construct a satisfying truth assignment for C'.

For variables in C' that are already in C, we use same truth assignments as for C.

How should we assign T/F to the new variables?
Truth Assignment for New Variables

Let $c_i = z_1 \vee z_2 \vee ... \vee z_k$

Case 1: $k = 1$. Use extra variables $y_{i1}$ and $y_{i2}$. Replace $c_i$ with 4 clauses:

$$(z_1 \vee y_{i1} \vee y_{i2}) \land (z_1 \vee \neg y_{i1} \vee y_{i2}) \land (z_1 \vee y_{i1} \vee \neg y_{i2}) \land (z_1 \vee \neg y_{i1} \vee \neg y_{i2}).$$

Assign $y_i$'s with arbitrary values, as $z_1$ is true.
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor ... \lor z_k$

Case 2: $k = 2$. Use extra variable $y_i^1$. Replace $c_i$ with 2 clauses:

$$(z_1 \lor z_2 \lor \neg y_i^1) \land (z_1 \lor z_2 \lor y_i^1).$$

Assign $y_i$'s with arbitrary values, as $z_1 \lor z_2$ is true
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 3: $k = 3$. No extra variables are needed.

Keep $c_i$: $(z_1 \lor z_2 \lor z_3)$
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 4: $k > 3$. Use extra variables $y_{i,1}^1, \ldots, y_{i,k-3}^{k-3}$. Replace $c_i$ with $k-2$ clauses:

$$(z_1 \lor z_2 \lor y_{i,1}^{1})$$

$$\land (-y_{i,1}^{1} \lor z_3 \lor y_{i,2}^{2}) \land (-y_{i,2}^{2} \lor z_4 \lor y_{i,3}^{3}) \land \ldots$$

$$\land (-y_{i,k-5}^{k-5} \lor z_{k-3} \lor y_{i,k-4}^{k-4}) \land (-y_{i,k-4}^{k-4} \lor z_{k-2} \lor y_{i,k-3}^{k-3})$$

$$\land (-y_{i,k-3}^{k-3} \lor z_{k-1} \lor z_k)$$

If $z_1$ or $z_2$ is true, set all $y_i$'s to false, so all later clauses have a true literal.
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor ... \lor z_k$

Case 4: $k > 3$. Use extra variables $y_i^1, ..., y_i^{k-3}$. Replace $c_i$ with $k-2$ clauses:

$$(z_1 \lor z_2 \lor y_i^1)$$

$$\land (\neg y_i^1 \lor z_3 \lor y_i^2) \land (\neg y_i^2 \lor z_4 \lor y_i^3) \land ...$$

$$\land (\neg y_i^{k-5} \lor z_{k-3} \lor y_i^{k-4}) \land (\neg y_i^{k-4} \lor z_{k-2} \lor y_i^{k-3})$$

$$\land (\neg y_i^{k-3} \lor z_{k-1} \lor z_k)$$

If $z_{k-1}$ or $z_k$ is the first true literal of $c_i$, set all $y_i$'s to true, so all earlier clauses have a true literal.
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 4: $k > 3$. Use extra variables $y_{i1}, \ldots, y_{i(k-3)}$. Replace $c_i$ with $k-2$ clauses:

\[
(z_1 \lor z_2 \lor y_{i1}) \land (\neg y_{i1} \lor z_3 \lor y_{i2}) \land (\neg y_{i2} \lor z_4 \lor y_{i3}) \land \ldots \land (\neg y_{i(k-5)} \lor z_{k-3} \lor y_{i(k-4)}) \land (\neg y_{i(k-4)} \lor z_{k-2} \lor y_{i(k-3)}) \land (\neg y_{i(k-3)} \lor z_{k-1} \lor z_k)
\]

If first true literal is in between, set all earlier $y_i$'s to true and all later $y_i$'s to false.
Correctness of Reduction

\[\leq: \text{Suppose the newly constructed 3SAT formula } C' \text{ is satisfiable. We must show that the original SAT formula } C \text{ is also satisfiable.}\]

Use the same satisfying truth assignment for } C \text{ as for } C' \text{ (ignoring new variables).}

Show each original clause has at least one true literal in it.
Let \( c_i = z_1 \lor z_2 \lor \ldots \lor z_k \)

Case 1: \( k = 1 \). Use extra variables \( y_i^1 \) and \( y_i^2 \). Replace \( c_i \) with 4 clauses:

\[
c_i' = (z_1 \lor y_i^1 \lor y_i^2) \land (z_1 \lor \neg y_i^1 \lor y_i^2) \land (z_1 \lor y_i^1 \lor \neg y_i^2) \land (z_1 \lor \neg y_i^1 \lor \neg y_i^2).
\]

If \( c_i' \) is true, then \( c_i = z_1 \) must be true, since one pair of literals in \( y_i^1 \) and \( y_i^2 \) must be true.
Reduction from SAT to 3SAT

Let \( c_i = z_1 \lor z_2 \lor \ldots \lor z_k \)

Case 2: \( k = 2 \). Use extra variable \( y_i \). Replace \( c_i \) with 2 clauses:

\[
c_i' = (z_1 \lor z_2 \lor \neg y_i) \land (z_1 \lor z_2 \lor y_i).
\]

If \( c_i' \) is true, then \( c_i = z_1 \lor z_2 \) must be true.
Reduction from SAT to 3SAT

Let $c_i = z_1 \lor z_2 \lor \ldots \lor z_k$

Case 3: $k = 3$. No extra variables are needed.

Keep $c_i$: $(z_1 \lor z_2 \lor z_3)$
Reduction from SAT to 3SAT

Let \( c_i = z_1 \lor z_2 \lor ... \lor z_k \)

**Case 4:** \( k > 3 \). Use extra variables \( y_{i1}, ..., y_{i(k-3)} \). Replace \( c_i \) with \( k-2 \) clauses:

\[
(z_1 \lor z_2 \lor y_{i1}) \land (\neg y_{i1} \lor z_3 \lor y_{i2}) \land (\neg y_{i2} \lor z_4 \lor y_{i3}) \land ...
\land (\neg y_{i(k-5)} \lor z_{k-3} \lor y_{i(k-4)}) \land (\neg y_{i(k-4)} \lor z_{k-2} \lor y_{i(k-3)})
\land (\neg y_{i(k-3)} \lor z_{k-1} \lor z_k)
\]

Suppose that there is a valuation such that \( c_i' \) is true and \( c_i \) is false. Then \( y_{ik} \) must be true for all \( k \), so the last clause in \( c_i' \) must be false, contradiction.
Conclusions

We have shown that

- 3SAT is in NP
- there exists a polynomial time reduction from SAT to 3SAT.

Therefore, 3SAT is NP-complete.