

## Useful Notation

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When we discussed geometric random variables, we encountered the equality

$$\sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \Pr[X = j] = \sum_{j=1}^{\infty} \sum_{i=1}^j \Pr[X = j], \quad (1)$$

which appears to be a bit puzzling. It turns out that with the proper notation, the manipulation of such sums becomes a routine matter.

Given a predicate *stmt*, we follow Iverson and use the bracket notation

$$[stmt] = \begin{cases} 1 & \text{if } stmt \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

The bracket notation allows us to write summations without explicit bounds. As a simple example, let us convince ourselves that equation (1) is correct. We have

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \Pr[X = j] &= \sum_i \sum_j \Pr[X = j] [1 \leq i < \infty][i \leq j \leq \infty] \\ &= \sum_j \sum_i \Pr[X = j] [1 \leq i \leq j < \infty] \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^j \Pr[X = j]. \end{aligned} \quad (2)$$

This notation makes such summation manipulations quite transparent. Let us have a look at another example. In the analysis of randomized quicksort, we encountered the equality

$$\sum_{i=1}^n \sum_{k=2}^{n-i+1} \frac{2}{k} = \sum_{k=2}^n \sum_{i=1}^{n+1-k} \frac{2}{k}. \quad (3)$$

Using the Iverson notation, one can explain this equality without hassle. Indeed,

$$\begin{aligned} \sum_{i=1}^n \sum_{k=2}^{n-i+1} \frac{2}{k} &= \sum_i \sum_k \frac{2}{k} [1 \leq i \leq n][2 \leq k \leq n - i + 1] \\ &= \sum_i \sum_k \frac{2}{k} [1 \leq i \leq n - 1][2 \leq k \leq n - i + 1] && i \text{ cannot attain } n \\ &= \sum_i \sum_k \frac{2}{k} [2 \leq n - i + 1 \leq n][2 \leq k \leq n - i + 1] && \text{permute range of } i \\ &= \sum_i \sum_k \frac{2}{k} [2 \leq k \leq n - i + 1 \leq n] && \text{combine statements} \\ &= \sum_k \sum_i \frac{2}{k} [2 \leq k \leq n][1 \leq i \leq n + 1 - k] && \text{split into two parts} \\ &= \sum_{k=2}^n \sum_{i=1}^{n+1-k} \frac{2}{k}. \end{aligned}$$

For more details on the manipulation of sums, see [Graham, Knuth, Patashnik, *Concrete Mathematics*, Addison Wesley, 2nd edition, 1994].