Generalized Second Price Auction (GSP) & Sponsored Search

In 2005, Google’s total revenue was $61 billion, over 98% of which came from online advertising; Yahoo!’s revenue was $5.3 billion, much of which also came from sponsored search.

Sponsored search was introduced as a new model for selling online ads in 1997 by Overture, at the time called GoTo, a company subsequently acquired by Yahoo.

In the original model, each advertiser submitted a bid reporting the advertiser’s willingness to pay for a click on its ad, for a particular keyword. Based on their bids, advertisers are allocated advertising slots in descending order on the right of the search results page.

In the original model, advertisers allocated a slot paid their bid (Generalized First Price Auction), however this mechanism was unstable since advertisers changed bids frequently in search of the lowest bid that could guarantee them a given position.

As a result, Google switched to using the GSP Auction, which proved more stable, and Yahoo! followed. Google erroneously advertised that its auction model is truthful since it generalizes Vickrey’s second price auction, in which truth-telling is a dominant strategy. However, GSP is different from the strategyproof VCG auction for this setting.
### Example 1: Generalized First Price Auction

<table>
<thead>
<tr>
<th>Advertisers</th>
<th>Values per click</th>
<th>Bids</th>
<th>Prices</th>
<th>2 slots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$10</td>
<td>$b_1</td>
<td>$b_1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$4</td>
<td>$b_2</td>
<td>$b_2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$2</td>
<td>$b_3</td>
<td>$b_3</td>
<td></td>
</tr>
</tbody>
</table>

There is no pure strategy equilibrium here.

Suppose $b_2 = $2.01 to guarantee winning a slot.

=> $b_1 = $2.02

=> $b_2 = $2.03

=> $b_1 = $2.04

shows one example of cycling behavior (as occurred in practice).

=> If advertisers best-respond to each other, they will be revising their bids as often as possible.

### Example 2: Generalized Second-Price Auction (GSP)

<table>
<thead>
<tr>
<th>Advertisers</th>
<th>Values</th>
<th>Bids</th>
<th>Prices</th>
<th>2 slots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>10</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In this example, truth-telling is a dominant strategy, because no advertiser benefits from changing his bid.

(total payments are $800 for adv. 1, $200 for adv. 2)
GSP and VCG

GSP looks similar to VCG because the prices that advertisers pay do not depend on their own bid, only on the bids of others.

With only one slot, GSP and VCG are indeed identical, being simply the second-price auction for selling a single item (in which truth-telling is a dominant strategy!).

However, with more than one slot, they are different: GSP charges the advertiser in position $i$ the bid of the advertiser in position $i+1$.

VCG charges the advertiser in position $i$ the externality he imposes on others by taking one of the slots away from them; this externality is equal to the difference between the aggregate value of clicks that all others would have received if advertiser $i$ were not present in the market and the aggregate value of clicks that all others would have received if $i$ were present.

Advertiser $j < i$ (labelling advertisers by their slot position) is not affected by $i$ $\Rightarrow$ externality on $j$ is 0.

Advertiser $j > i$ would have received position $(j-1)$ in absence of $i$ $\Rightarrow$ externality on $j > i$ is given by her value per click $v_j \cdot (\text{CTR}_{j-1} - \text{CTR}_j)$

$\text{CTR}_j =$ Click-Through Rate of slot $j$ (# clicks a slot is expected to receive in a given time period).
Example 3: VCG Payments

<table>
<thead>
<tr>
<th>Advertiser</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$v_1 = 10$</td>
</tr>
<tr>
<td>2</td>
<td>$v_2 = 4$</td>
</tr>
<tr>
<td>3</td>
<td>$v_3 = 2$</td>
</tr>
</tbody>
</table>

Advertiser 2 imposes \(\) no externality on advertiser 1, \(\) externality = \(200 \times CTR_2\) on advertiser 3

\[
=> \text{Advertiser 2's payment under VCG is 200.}
\]

Advertiser 1 imposes \(\) externality = \(200\) on advertiser 3
\(\) externality = \(400 = v_2 \cdot (CTR_1 - CTR_2)\) on advertiser 2

\[
=> \text{Advertiser 1's payment under VCG is 600 (compare to 800 in GSP from Example 2).}
\]

Formal Model of GSP

Given keyword
\(N\) slots
\(K\) bidders (advertisers)
\(\alpha_i\) = expected # clicks per period from slot \(i\)

\(s_k\) = value per click to advertiser \(k\)
(Previously denoted \(v_k\)).

Advertiser \(k\)'s utility from slot \(i\) is

\[
U_k = \alpha_i s_k - \text{payment}_k
\]

Positions are labelled in descending order: \(\alpha_i > \alpha_j\)

\(b_k\) = advertiser \(k\)'s bid for given keyword

\(b(j)\) = bid of \(j^{th}\) highest bidder

\(g(j)\) = identity of \(j^{th}\) highest bidder (ties are ordered randomly)
Mechanism allocates top position to highest bidder \( g(1) \) second — \( g(2) \) — second highest bidder \( g(2) \) ...

\[
\min \{ N, K \} \text{ position to corresponding bidder.}
\]

**Payments in GSP**

Advertiser's payment per click = next advertiser's bid

\[
\left( g(j) \right) \text{'s payment is } \frac{b^{(j+1)}}{b^{(j+1)}}. \]

\[
\Rightarrow g(j) \text{'s total payment } p^{(j)} = \sum_{j=1}^{\min \{ N, K \}} \frac{b^{(j+1)}}{b^{(j+1)}}. \]

\[
\mu_{g(j)} = \lambda_j \left( S_{g(j)} - b^{(j+1)} \right). \]

If \( N \geq K \) ( slots \( \geq \) advertisers)

\[
\Rightarrow \text{last slot costs zero. ( } p^{(K)} = 0. \)

**Payments in VCG**

VCG allocates bidders in same way as GSP.

Each advertiser's payment = externality he imposes on others, given assuming that bids = values.

\[
\Rightarrow \text{Payment for last advertiser allocated a slot is same as in GSP = } \begin{cases} 0 & \text{if } N \geq K \\ \frac{\lambda_j b^{(N+1)}}{N+1} & \text{if } N < K \end{cases}
\]

For all other \( i < \min \{ N, K \} \):

\[
P_{\text{VCG}}(i) = (\lambda_i \cdot \lambda_{i+1}) b^{(i+1)} + P_{\text{VCG}}(i+1)
\]

**Remark 1:** If all advertisers bid same amounts under two mechanisms, then each advertiser's payment would be at least as large under GSP as under VCG.

**Pf:** By induction on advertisers' payments, starting with last one who gets a slot.
(Proof of remark 1 continued):

For \( i = \min \{ K, N \} \), \( p^{(i)} = v_{\text{VCG}}^{(i)} = a_i b^{(i+1)} \)

For \( i < \min \{ K, N \} \),

\[
\begin{align*}
p^{\text{VCG}}(i) - p^{\text{VCG}}(i+1) &= (a_i - a_{i+1}) b^{(i+1)} \\
&\leq a_i b^{(i+1)} - a_{i+1} b^{(i+2)} \\
&= p^{(i)} - p^{(i+1)}
\end{align*}
\]

Remark 2: Truth-telling is a dominant strategy under VCG.

Remark 3: Truth-telling is not a dominant strategy under GSP.

Example 4:

<table>
<thead>
<tr>
<th>Advertiser</th>
<th>Values</th>
<th>Bids</th>
<th>Per click</th>
<th>2 Slots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>( b_1 )</td>
<td>( b_2 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>( b_2 )</td>
<td>( b_3 )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>( b_3 )</td>
<td></td>
<td>200 clicks/hr</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>199 clicks/hr</td>
</tr>
</tbody>
</table>

If advertisers bid their true values:

Utility of advertiser 1, \( u_1 = (10 - 4) \times 200 = 1200 \)

If instead \( b_1 = 3 \) so he gets slot 2, with price 2/

\[
\Rightarrow u_1 = (10 - 2) \times 199 = 1592 > 1200.
\]
If vector of bids stabilizes, at what bids can it stabilize? Model auction as simultaneous move game with all values being common knowledge (reasonable to assume after having played for a long period of time).

Assuming bids form equilibrium in simultaneous-move one-shot game of complete information, what are simple strategies an advertiser can use to increase his payoff, beyond simple best responses to others' bids?

Eg: force out the bidder in position immediately above

Suppose advertiser \( k \) bidding \( b_k \) is in slot \( i \)

\[ k' \rightarrow b_k \rightarrow (i-1) \]

If \( k \) raises his bid slightly, his position and payment do not change but utility of bidder \( k' \) above decreases.

\( k' \) can retaliate, slightly underbidding \( k \)

If \( k \) is better off after retaliation, he will have successfully forced \( k' \) out and bids change.

\[ \Rightarrow \] If vector of bids has reached a stable position, advertiser in slot \( i \) should not want to be in slot \( (i-1) \)

\[ \rightarrow \] this vector of bids called "locally envy-free".

**Def:** An equilibrium of the simultaneous-move game induced by GSP is **locally envy-free** if a player cannot improve his payoff by exchanging bids with the player ranked one position above him, i.e. for all \( i \leq \min \{ N, k \} \),

\[ \alpha_i S_g(i) - P(i) \geq \alpha_{i-1} S_g(i) - P(i-1) \]
The following is a locally envy-free equilibrium of GSP:
Assume advertisers are labelled by decreasing order of value, 
\(j < k \) if \( s_j > s_k \).
For each advertiser \( j \in \{2, \ldots, \min\{N+1, K\}\} \),
\[
\text{bid } b_j^* = \frac{p_{\text{VCG}(j-1)}}{\alpha_{j-1}},
\]
where \( p_{\text{VCG}(j-1)} \) is the payment of advertiser \((j-1)\) in
the truthful dominant strategy equilibrium of VCG.

For advertiser 1,
\[
b_1^* = s_1 \quad (\text{here, anything greater than } b_2^* \text{ will work}).
\]

**Theorem:** Strategy profile \((b_1^*, \ldots, b_K^*)\) is a
locally envy-free equilibrium of the game. In this
equilibrium, each advertiser's position and payment
are equal to those in the truthful dominant strategy
equilibrium of the game induced by VCG.
In any other locally envy-free equilibrium, the
total revenue of the seller is at least as high
as in \((b_1^*, \ldots, b_K^*)\).

**References:** These notes are based on
"Internet Advertising and the Generalized Second-Price
Auction: Selling Billions of Dollars Worth of Keywords"
by Edelman, Ostrovsky & Schwarz. (American Economic
Review, 2007)