Lecture 2

Two-player strategic form games.

A two-player strategic form game is specified via a pair \((R, C)\) of \(m \times n\) payoff matrices. The row player (player 1) has \(m\) pure strategies, corresponding to the \(m\) rows. The column player (player 2) has \(n\) pure strategies, corresponding to the \(n\) columns.

If the row player plays row \(i\) and the column player plays column \(j\), their payoffs are \(R_{i,j}\) and \(C_{i,j}\) respectively.

\[
\begin{array}{c|cc}
   & C_1 & C_2 \\
\hline
R_1 & R_{11}, C_{11} & R_{12}, C_{12} \\
R_2 & R_{21}, C_{21} & R_{22}, C_{22} \\
R_3 & R_{31}, C_{31} & R_{32}, C_{32} \\
\end{array}
\]

eg: With slight abuse of notation, denote actions of player 1 by \(R_1, \ldots, R_m\), player 2 by \(C_1, \ldots, C_n\).

A randomized (mixed) strategy for row player is a vector \(x = \left( x_1, \ldots, x_m \right)\) with of probabilities \(0 \leq x_i \leq 1\) so that \(\sum_{i=1}^{m} x_i = 1\). Denote the simplex \(\Delta_m\), the set of all such feasible vectors in dimension \(m\). Similarly, denote a randomized (mixed strategy for column player by \(y \in \Delta_n\).
In the two-player setting:

\[ \mathcal{S}_1 = m \text{ actions of row player (player 1)} \]
\[ \mathcal{S}_2 = n \text{ actions of column player (player 2)} \]

Denote the mixed strategies of player 1, 2 by

\[ x \in \Delta_m, \quad y \in \Delta_n \text{ respectively, where} \]
\[ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \]

**def:** A pair of strategies \((x, y)\) is a Nash Equilibrium if and only if neither player can increase her payoff by unilaterally deviating from her strategy:

\[ x^T R y \geq x'^T R y, \quad \forall x' \in \Delta_m \]
\[ x^T C y \geq x'^T C y', \quad \forall y' \in \Delta_n \]

**Note:** We assume that the payoff to player 1 from playing mixed strategy \(x\) is linear in her expected utility. When player 2 plays the mixed strategy \(y\),

\[ u_1(x, y) = \sum_{i=1}^{m} x_i u_i(R_i, y) \]
\[ = \sum_{i=1}^{m} x_i \left( \sum_{j=1}^{n} y_j \cdot u_i(R_i, c_j) \right) \]
\[ = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j r_{ij} = x^T R y \]

Similarly for player 2.

By linearity of expectation, the support of a player's mixed strategy should only include pure strategies that maximize her payoff, given other player's strategy.
(Otherwise, supposing on the contrary that the support includes pure strategy \( R_i \) with \( u_i(R_i, y) < \max_{1 \leq k \leq m} u_k(R_k, y) \), transferring all probability weight \( x_i \) from \( R_i \) to the maximizing pure strategy will increase the expected payoff, a contradiction.)

Alternative NE definitions:

\[ \text{def 2: } \begin{align*} &i \text{ with } x_i > 0, \quad i \in \arg \max_k \{ e_k^T R y \}, \\ &j \text{ with } y_j > 0, \quad j \in \arg \max_k \{ x^T C e_j \} \end{align*} \]

where \( e_k \in \Delta_m \) is the vector with all zeroes and a single 1 at coordinate \( k \), similarly for \( e_j \in \Delta_m \),

\( \rightarrow \) these are pure strategies.

\[ \text{def 3: } \begin{align*} &x^T R y \geq e_i^T R y, \quad i \\ &x^T C e_j \geq x^T C e_j, \quad j \end{align*} \]

where \( e_i \) represents pure strategy \( i \) of player 1 \( e_j \) \( j \) of player 2.
Presidential Elections:

\[
\begin{array}{c|cc|c}
   & y_1 & y_2 & \text{Zero-sum game!} \\
\hline
x_1 (E) \text{ Economy} & 3, -3 & -1, 1 & i.e. R_{ij} + C_{ij} = 0 \\
x_2 (S) \text{ Society} & -2, 2 & 1, -1 & \forall i,j
\end{array}
\]

Let's find the Nash Equilibria (NE)

Check that there are no pure strategy NE.

Finding mixed equilibria:

Approach 1: Suppose player 2 plays \((y_1, y_2)\), \(y_1 + y_2 = 1\)

Player 1 payoff:

\((E)\) : \(3y_1 - y_2\)

\((S)\) : \(-2y_1 + y_2\)

So if \(3y_1 - y_2 > -2y_1 + y_2\) \(\Rightarrow\) Player 1 plays \((E)\)

\(\iff\) \(5y_1 > 2y_2\) \(\Rightarrow\) Player 2 plays \((T)\)

\(\iff\) \(5y_1 > 2(1-y_1)\) \(\iff\) \(y_1 = 0\)

\(\iff\) \(5y_1 > 2-2y_1\)

\(\iff\) \(7y_1 > 2\) \(\iff\) \(y_1 > \frac{2}{7}\) \(\leftarrow\) contradiction.

If \(3y_1 - y_2 < -2y_1 + y_2\) \(\Rightarrow\) Player 1 plays \((S)\)

\(\iff\) \(y_1 < \frac{2}{7}\) \(\Rightarrow\) Player 2 plays \((M)\)

\(\iff\) \(i.e. \quad y_1 = 1\)

\(\iff\) \(y_2 = 0\)

\(\leftarrow\) contradiction.

If \(3y_1 - y_2 = -2y_1 + y_2\) \(\Rightarrow\) Player 1 indifferent

\(\iff\) \(y_1 = \frac{2}{7}\)

\(\leftarrow\) between \((E), (S)\)
Suppose player 1 plays \((x_1, x_2)\), \(x_1 + x_2 = 1\)

Player 2 payoff:

\[(M) : -3x_1 + 2x_2\]

\[(T) : x_1 - x_2\]

If \(-3x_1 + 2x_2 > x_1 - x_2\) \(\Rightarrow\) Player 2 plays \((M)\)

\[\Leftrightarrow 3x_2 > 4x_1\]

\[\Leftrightarrow 3(1-x_1) > 4x_1\]

\[\Leftrightarrow 7x_1 < 3\]

\[\Leftrightarrow x_1 < \frac{3}{7}\] contradiction.

If \(-3x_1 + 2x_2 < x_1 - x_2\) \(\Rightarrow\) player 2 plays \((T)\)

\[\Leftrightarrow x_1 > \frac{3}{7}\] \(\Rightarrow\) Player 1 plays \((S)\)

\[\text{i.e. } x_1 = 0, x_2 = 1\] contradiction.

If \(-3x_1 + 2x_2 = x_1 - x_2\) \(\Rightarrow\) Player 2 indifferent

\[\Leftrightarrow x_1 = \frac{3}{7}\]

between \((T), (S)\)

Conclusion: Unique NE is

\[x = \left( \frac{3}{7}, \frac{4}{7} \right), \quad y = \left( \frac{2}{7}, \frac{5}{7} \right)\]

with expected payoffs:

\[3y_1 - y_2 = \frac{1}{7}, \quad \frac{-1}{7}\]
(Approach 2): Suppose players 1, 2 play \( x, y \) respectively.

If row player announces \( x \) in advance

\[
\Rightarrow \text{column player's optimal play yields}
\]

\[
\max \left\{ -3x_1 + 2x_2 \right\} \quad \Rightarrow \quad x_1 - x_2
\]

Zero-sum game \( \Rightarrow \) row player's payoff is

\[
- \max \left\{ -3x_1 + 2x_2 \right\} = \min \left\{ 3x_1 - 2x_2, -x_1 + x_2 \right\}
\]

\[
\Rightarrow \text{Optimal player 1 strategy (in advance) is}
\]

\[
(x_1, x_2) = \arg \max_{(x_1, x_2)} \min \left\{ 3x_1 - 2x_2, -x_1 + x_2 \right\}
\]

which can be computed via a linear program (LP):

\[
\begin{align*}
\text{max} & \quad Z \\
\text{s.t.} & \quad 3x_1 - 2x_2 \geq Z \\
& \quad -x_1 + x_2 \geq Z \\
& \quad x_1 + x_2 = 1 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Solution: \( x_1 = \frac{3}{7}, \; x_2 = \frac{4}{7}, \; Z = \frac{1}{7} \).

Similarly, optimal player 2 strategy (in advance) solves

\[
\begin{align*}
\text{max} & \quad W \\
\text{s.t.} & \quad -3y_1 + y_2 \geq W \\
& \quad 2y_1 - y_2 \geq W \\
& \quad y_1 + y_2 = 1 \\
& \quad y_1, y_2 \geq 0
\end{align*}
\]

Solution: \( y_1 = \frac{2}{7}, \; y_2 = \frac{5}{7}, \; W = -\frac{1}{7} \).
Observations: If player 1 plays \( x = \left( \frac{3}{7}, \frac{4}{7} \right) \)
\[-1\] 2 plays \( y = \left( \frac{2}{7}, \frac{5}{7} \right) \), then:

1) Payoff of player 1 is \( \frac{1}{7} \) \( \sum \) payoffs = 0
\[-1\]
\[-2\] is \(-\frac{1}{7}\)

2) Given \( x = \left( \frac{3}{7}, \frac{4}{7} \right) \) for player 1,
player 1 gets at least payoff \( \frac{1}{7} \), regardless of player 2's strategy.
player 2 gets at most \(-\frac{1}{7}\), by zero sum.

3) Given \( y = \left( \frac{2}{7}, \frac{5}{7} \right) \) for player 2,
player 2 gets \( \geq -\frac{1}{7} \) regardless of player 1's strategy.
\( \Rightarrow \) Player 1 gets \( \leq \frac{1}{7} \).

\( \Rightarrow \) \( x = \left( \frac{3}{7}, \frac{4}{7} \right), \ y = \left( \frac{2}{7}, \frac{5}{7} \right) \) is NE.

Surprising? Suboptimal / pessimistic to announce strategy in advance of other player.
Why would the two pessimistic strategies be best responses to each other?

LP Duality
Nash Equilibria and LP Duality

(2-player zero-sum games: \( R + C = 0 \)).

As above, if row player must announce \( x \) in advance, optimal \( x \) solves:

\[
\begin{align*}
\text{LP (1)}: & \quad \max \ z \\
\text{s.t.} & \quad x^T R \geq z^T \mathbf{1}^T \quad \text{vector of ones} \\
& \quad x^T \mathbf{1} = 1 \\
& \quad x \geq 0
\end{align*}
\]

Maximum of \( z = \max_x \min_y x^T R y \)

( Remember, when row player plays \( x \), column player plays \( y \), payoff to column player is \( x^T C y \).

\( \Rightarrow \) column player should \( \max_y x^T C y \)

\( \Rightarrow \) Row player would get \( -\max_y x^T C y = \min_y x^T R y \) )

Dual LP: to LP (1):

\[
\begin{align*}
\text{LP (2)}: & \quad \min \ z' \\
\text{s.t.} & \quad -y^T R + z' \mathbf{1}^T \geq 0 \\
& \quad y^T \mathbf{1} = 1 \\
& \quad y \geq 0
\end{align*}
\]

Let \( z'' = -z' \) \( \Rightarrow \) \( c = -R \)

\( \text{LP (3)}: \)

\[
\begin{align*}
\text{LP (2)} & \iff \max \ z'' \\
\text{s.t.} & \quad C y \geq z'' \mathbf{1} \\
& \quad y^T \mathbf{1} = 1 \\
& \quad y \geq 0
\end{align*}
\]

\[
\max z'' = \max_y \min_x x^T C y = -\max_{y,x} x^T R y
\]
Similarly to LP(1), we can see that
\[
\text{maximum of } z'' = \max_y \min_x x^T C y = \max_y \min_x x^T R y
\]

Note, LP(3) is what column player would solve if she needs to announce her strategy \( y \) in advance.

Next: using strong duality of LP(1), LP(3) we'll show that solutions to LP(1), LP(3) yield a NE.

\[\begin{align*}
\text{LP Strong Duality} \quad & \quad \text{by LP} \\
\text{If } (x, z) \text{ is optimal for LP}(1) \} & \Rightarrow z = z', \\
(y, z') \text{ is optimal for LP}(2) & \text{duality} \\
\text{So if } (x, z) \text{ opt. for LP}(1) \} & \Rightarrow z = -z''.
\end{align*}\]
Theorem 1: If \((x, z)\) is optimal for LP(1)
\((y, z'')\) is optimal for LP(3)
\(\Rightarrow (x, y)\) is NE of 2-player zero-sum \((R, C)\)

Furthermore, the payoffs of player 1, 2 are \(z, z'' = -z\) respectively.

\[\text{Pf: } (x, z) \text{ is feasible for LP(1), } (y, z'') \text{ is feasible for LP(3)}\]
\[
\begin{align*}
\Rightarrow x^TR \geq z'1^T & \Rightarrow x^TRy \geq z + ty \\
\Rightarrow cy \geq z''1 & \Rightarrow x^TCy \geq z'' + tx' \\
\Rightarrow x^TRy \leq -z'' + tx' & \end{align*}
\]

If player 2 plays \(y\) \(\Rightarrow\) from LP(3)
player 2's payoff is \(\geq z'' \) \(\Rightarrow\) \(tx'\)
\(\Rightarrow\) player 1's payoff is \(\leq -z'' = z\)

If player 1 plays \(x\) \(\Rightarrow\) from LP(1), her payoff is \(z\).
\(\Rightarrow x\) is a best response to \(y\).

Similarly, \(y\) is a best response to \(x\), QED.
And payoffs are \(z, -z = z''\) for players 1, 2 respectively.

(There exists)

Corollary 1: \(\exists\) NE in every 2-player zero-sum game

Corollary 2: \([\text{THE MIN MAX THEOREM}]\) \[
\begin{align*}
\max_x \min_y x^TRy &= \min_y \max_x x^TRy \\
\end{align*}
\]
**Theorem 2:** If \((x, y)\) is NE of game \((R, C)\), then \((x, x^T R y)\) is an optimal soln of \(LP(1)\) and \((y^*, -x^T R y)\) is an optimal soln of \(LP(2)\).

**Pf:** Denote \(z = x^T R y = -x^T C y\).

\((x, y)\) is NE \(\Rightarrow x^T C y \geq x^T C e_j\) \(\forall j\)
\(\Rightarrow -x^T R y \geq -x^T R e_j\) \(\forall j\)
\(\Rightarrow x^T R e_j \geq z\)
\(\Rightarrow x^T R \geq z \cdot 1\)

\((x, z)\) is a feasible solution for \(LP(1)\).

Similarly, \((y, z)\) is a feasible solution for \(LP(2)\).

By weak duality, the value of a feas. soln of \(LP(2)\)
\[\geq\] value of a feas. soln of \(LP(1)\)

And if these values are equal, the feasible solns are optimal for the two LPs:

\((x, z)\) is optimal for \(LP(1)\)

\((y, z)\) is optimal for \(LP(2)\)

\(\Rightarrow (y, -z)\) is optimal for \(LP(3)\), QED

\(\Rightarrow\) All NE are pairs of optimal solns to \(LP(1), LP(2)\) and vice versa.

**Corollary 3:** (Convexity of Optimal Strategies): The set \(\mathcal{E}\) of equilibrium strategies for row player
\[
\mathcal{E} = \{x \mid \text{there exists } y \text{ s.t. } (x, y) \text{ is NE of } (R, C) \}
\]

is convex. Same for column player.

**Cor. 4:** (Anything goes). If \(x\) is any equil. strategy for row player and \(y\) is any equil. strategy for col. player \(\Rightarrow (x, y)\) is NE.
Corollary 5: (Convexity of Equil. Set) The set of NE \{ (x, y) \mid (x, y) \text{ is NE of } (R, C) \} of a zero-sum game is convex.

Corollary 6: (Uniqueness of payoffs) The payoff of row (column) player in all NE is same. Similarly for the column player.

It can define the (value) of a zero-sum game:

Def. (Value of Zero-Sum Game). If \((R, C)\) is a zero-sum game, the value of the game is the unique payoff of the row player in all NE of the game.