VCG Mechanism for Interdomain Routing

- Given a network with $N$ nodes, denote node set $N$.
- Each node represents an AS (Autonomous system / Internet domain).
- Assume biconnected (at least two potential routes between any pair of nodes).
- For each pair of nodes $(i, j)$, $T_{ij}$ is the intensity of traffic (number of packets) from $i$ to $j$.
- Node $k$ incurs transit cost $C_k$ for each transit packet it carries.

In Mechanism Design terminology, $C_k$ is $k$'s type.

Assume $C_k$ is independent of which packet node $k$ received packet from and which neighbor $k$ sent packet to.

[The approach here can be extended to differing costs; in this case, have costs associated with edges rather than the nodes; the strategic agents would still be nodes.]

Cost vector $C := (C_1, \ldots, C_N)$ of all transit costs $C^{-k} := (C_1, \ldots, C_{k-1}, C_{k+1}, \ldots, C_N)$: all costs except $C_k$.

- Each node $k$ is given a payment $p_k$ to compensate it for carrying transit traffic.
- Assumption: Nodes that carry no traffic receive no payment.

**GOAL**: Send each packet along least cost path (LCP) with respect to the true costs $C$. 
Denote by \( I_k(c; i, j) \) the indicator function of whether node \( k \) is on the LCP from \( i \) to \( j \) or not (then, \( I_k(c; i, j) = 1 \) or 0 respectively). We only count transit nodes \( k \), so that
\[
I_i(c; i, j) = I_j(c; i, j) = 0.
\]

When traffic is routed along LCPs, node \( k \) incurs a cost \( c_k \) for a packet sent from \( i \) to \( j \) if and only if \( k \in \text{LCP}(i, j) \).

The total cost of node \( k \) is:
\[
U_k(c) = c_k \sum_{i, j \in \text{N}} T_{ij} I_k(c; i, j).
\]

We want to minimize the total system cost \( V(c) \) (total cost to society) of routing all packets:
\[
V(c) = \sum_k U_k(c) = \sum_i \sum_{j \in \text{N}} T_{ij} \sum_{k \in \text{N}} I_k(c; i, j) c_k.
\]

Minimizing \( V(c) \) is equivalent to minimizing the path cost between \( i, j \) for all node pairs \((i, j)\).

**Challenge:** \( V(c) \) is defined in terms of true costs, but routing algorithm operates on declared costs. Since nodes' transit costs are private, we must rely on a pricing scheme to incentivize agents to report their true costs.

**Solution:** Algorithmic mechanism, which adapts the VCG mechanism to this routing problem.
Mechanism Design Set-up:

Input: the graph and the vector of declared costs C (we use C both for true and reported costs, to avoid cluttering notation; context will clarify which meaning is used).

Output: 1) the set of least cost paths (LCP) → allocation
2) the set of prices

We are looking for a strategyproof mechanism so agents have no incentive to lie about their costs.

The utility of node k is its payment minus cost:

\[ \mathcal{U}_k(c) = p_k^k - \sum_{i,j \in N} T_{ij} I_k(c; i, j) c_k, \]

In this context, strategyproofness means that

\[ \mathcal{U}_k(c) \geq \mathcal{U}_k(c_1, \ldots, c_k, x, \ldots, c_n) \quad \text{for all } X \quad \text{the vector } (c_1, \ldots, c_{k-1}, x, c_{k+1}, \ldots, c_n) \]

Theorem: When routing picks lowest-cost paths, and the network is biconnected, there is a unique strategyproof pricing mechanism that gives no payment to nodes that carry no transit traffic. The payments to transit nodes are of the form

\[ p_k^k = \sum_{i,j \in N} T_{ij} p_{ij}^k, \quad \text{where} \]

\[ p_{ij}^k = c_k I_k(c; i, j) + \left[ \sum_{r \in N} I_r(c_1, \ldots, c_{k-1}, x, c_{k+1}, \ldots, c_n) \right] - \sum_{r \in N} I_r(c; i, j) c_r \]
Proof: By the Theorem of Green and Laffont (1979), proved in last lecture, any strategyproof mechanism that minimizes the social cost \( V(c) = \sum_{k \in \mathcal{C}} u_k(c) \) (namely, it's efficient), must be a VCG mechanism with payments of the form
\[
p^k = u_k(c) - V(c) + h_k(c^{-k}),
\]
where \( h_k \) is an arbitrary function of \( c^{-k} \).

[Recall from last lecture that Grove's transfers are:
\[
t_i(\theta) = \left[ \sum_{j \neq i} v_j(x^*(\theta), \theta_j) \right] + h_i(\theta - \theta_i)
\]

welfare of agent \( j \) from the efficient allocation \( x^*(\theta) \) when his type is \( \theta_j \).

When \( c_k = \infty \), \( I_k(c|^{\infty}_k; i,j) = 0 \) for all \( i,j \) because the graph is biconnected and all other costs are finite. \( \Rightarrow p^k = 0 \) and \( u_k(c) = 0 \)

\[
\Rightarrow h_k(c^{-k}) = V(c^*|^{\infty}_k)
\]

\[
p^k = V(c^{|^{\infty}_k}) + u_k(c) - V(c)
= u_k(c) + \underbrace{\left[ V(c^{|^{\infty}_k}) - V(c) \right]}_{\text{own cost}} \underbrace{\text{bonus equal to the positive externality node } k\text{'s presence brings to network}}_{\text{node } k\text{'s presence brings to network}}
= \sum_{i,j \in \mathcal{E}} T_{ij} \left[ c_k I_k(c; i,j) + \sum_{r \in \mathcal{N}} I_r(c|^{\infty}_k; i,j) c_r - \sum_{r \in \mathcal{N}} I_r(c; i,j) \right]
= \sum_{i,j \in \mathcal{E}} T_{ij} p^k_{ij}, \text{ where}
\]

\[
p^k_{ij} = c_k I_k(c; i,j) + \sum_{r \in \mathcal{N}} c_r I_r(c|^{\infty}_k; i,j) - \sum_{r \in \mathcal{N}} c_r I_r(c; i,j)
= \text{k's cost for packets on path } (i,j) \text{ bonus equal to the reduction in cost node } k\text{'s presence brings to the network for path } (i,j).\]
Remarks about mechanism:

1) Although payments could have taken any form and could have depended arbitrarily on traffic matrix, they turned out to be a sum of per-packet payments that do not depend on the traffic matrix.

2) The prices $P_{i,j}^k = 0$ if node $k$ is not on $LCP(i,j)$. So payments can be computed by counting packets as they enter node $k$, knowing the prices $P_{i,j}^k$.

3) The costs do not depend on the packets' source and destination but the prices do.

4) The payment to a node $k$ for a packet from $i$ to $j$ is determined by the cost of the LCP and the cost of the LCP that does not go through $k$. The difference between the two is the positive externality that node $k$'s presence brings to the network (the reduction in cost of the LCP when node $k$ is present, compared to its cost when node $k$ is absent).

Criticism of the VCG Mechanism: Overpayments

Suppose the costs of all nodes on top route are $e_i = 0$. Removing node $A_i$ results in the LCP through $B$ of cost 1, namely the cost of $LCP(S,T)$ increases by 1. Thus, each node $A_i$ must be paid its cost plus a bonus of 1, resulting in total payment > $n$. 