Equilibria in Risk-Averse Network Routing

Problem Definition

Consider a graph \( G = (V, E) \)

- \( K \) source-destination pairs \( (s_k, t_k), k = 1, \ldots, K \)

Demand (flow) \( d_k \) between \( s_k, t_k \), \( k = 1, \ldots, K \)

\( P_k = \) set of all paths between \( s_k, t_k \)

\( P = \bigcup_{k=1}^{K} P_k \) - set of all feasible paths

- Stochastic edge delays modeled as \( l_e(f_e) + \xi_e(f_e) \) for each edge \( e \in E \)

  - \( f_e \) is flow on edge \( e \)
  - \( |f_p| \) is flow on path \( p \in P \)

Sometimes, to distinguish between edge and path flow, we'll use \( x_e \) for edge flow.

Stochastic path delay for path \( p \in P \) is

\[
\frac{\sum_{e \in p} l_e(f_e)}{\text{path mean}} + \sqrt{\sum_{e \in p} \xi_e(f_e)} \quad \text{random variable with mean 0, st. dev. } \sqrt{\sum_{e \in p} \xi_e(f_e)}
\]

assuming \( \xi_e(f_e) \) are independent.

We will need an objective function — we will call it path cost — to represent the users’ risk aversion.
Path Cost Model

We will consider the mean-stdev path cost, defined as:

\[ Q_p(f) = \sum_{e \in p} l_e(f_e) + \delta \sqrt{\sum_{e \in p} \sigma_e^2(f_e)} \]

path mean

path standard deviation

coefficient of risk-aversion.

Pros of model:
- Widely used to represent risk-averse attitudes (e.g. finance; transportation practitioners)
- Interpretation under normal distributions:
  Equal to percentile of delay
  (the minimum time we need to budget for our trip to ensure on-time arrival with at least 95\%, say)

Cons:
- May result in stochastically dominated paths (not a monotone risk measure)
- Nonadditive, difficult to optimize (path cost ≠ \( \sum \) edge costs)

Def: Stochastic Wardrop Equilibrium: A flow \( f \) such that for all paths \( P_1, P_2 \in P_k \) with \( f_{P_1} > 0 \),

\[ Q_{P_1}(f) \leq Q_{P_2}(f) \]
Equilibrium existence & characterization

Recall that in the classical (deterministic) setting, the equilibrium solves a convex program:

\[
\begin{align*}
\text{min} & \quad \sum_{e \in E} h_e(f_e) \\
\text{s.t.} & \quad \sum_{p \in p_k} f_p = d_k, \\
\quad & \quad f_e = \sum_{p \in p_e} f_p, \\
\quad & \quad f_p \geq 0,
\end{align*}
\]

where \( h'_e(f_e) = l_e(f_e) \) so \( h_e(f_e) = \int_{f_e}^{f_e(x)} dx \) to get the equivalent first-order conditions:

\[
\frac{h'_{p_1}(f)}{l_{p_1}(f)} \leq \frac{h'_{p_2}(f)}{l_{p_2}(f)}
\]

We can also write above program equivalently in terms of path costs \( h_p(f) = \sum_{e \in p} h_e(f_e) \) :

\[
\begin{align*}
\text{min} & \quad \sum_{p \in P} h_p(f) \\
\text{s.t.} & \quad \sum_{p \in p_k} f_p = d_k, \\
\quad & \quad f_e = \sum_{p \in p_e} f_p, \\
\quad & \quad f_p \geq 0,
\end{align*}
\]

We will show that the stochastic routing setting has an equilibrium that solves a similar convex program, provided the standard deviations \( \sigma_e \) are constant (i.e., the uncertainty is exogenous, independent of the flow).

On the other hand, there is no convex program formulation when the uncertainty is endogenous (i.e., it depends on the flow).
Exogenous uncertainty

Path cost is \( Q_p(f) = \sum_{e \in e_p} le(f_e) + \gamma \sum_{e \in e_p} \sigma_e^2 \) constant!

The equilibrium condition \( Q_{p_1}(f) \leq Q_{p_2}(f) \) is a first-order condition for the program

\[
\begin{align*}
\min & \quad H(f) \\
\text{s.t.} & \quad \sum_{p \in P_k} f_p = d_k \\
& \quad fe = \sum_{p \in P} f_p \\
& \quad f_p \geq 0
\end{align*}
\]

provided

\[
\frac{\partial H(f)}{\partial f_p} = Q_p(f) \quad \text{for all } p \in P.
\]

Taking

\[
H(f) = \sum_{e \in e_p} \int_{t_e}^{t_f} le(x) \, dx + \gamma \sum_{p \in P} f_p \sqrt{\sum_{e \in e_p} \sigma_e^2}
\]

satisfies \((*)\).

\(H(f)\) is a convex function since both terms are convex (the second is linear in \(f_p\)'s)

\[\Rightarrow\] The equilibrium exists and is unique (provided \(H(f)\) is strictly convex).

Remark: In the deterministic setting, the convex program was of polynomial size and could be solved efficiently. Here, it is not: we have exponentially many variables \(f_p\) for all paths \(p \in P\). Computational complexity of finding equilibrium is open.
Denote \( f_1 = f_{\{S,A,T\}} \) → flow on \( \rightarrow p_1 \)
\( f_2 = f_{\{S,B,T\}} \) → flow on \( \rightarrow p_2 \)
\( f_3 = f_{\{S,A,B,T\}} \) → flow on \( \rightarrow p_3 \)

Suppose \( x^* = 1 \) and \( \sigma_e(f) = f \) ∀e
\[
\implies Q_1(f) = f_1 + f_3 + 1 + \sqrt{(f_1 + f_3)^2 + f_1^2}
\]
\[
Q_2(f) = f_2 + f_3 + 1 + \sqrt{(f_2 + f_3)^2 + f_2^2}
\]
\[
Q_3(f) = f_1 + f_2 + 2f_3 + \sqrt{(f_1 + f_3)^2 + f_3^2 + (f_2 + f_3)^2}
\]

Check that \( \frac{\partial Q_1(f)}{\partial f_3} = 1 + \frac{1}{\sqrt{(f_1 + f_3)^2 + f_1^2}} \)
\( \frac{\partial Q_3(f)}{\partial f_1} = 1 + \frac{1}{\sqrt{(f_1 + f_3)^2 + f_3^2 + (f_2 + f_3)^2}} \)

\( \implies \) Formulation (**) is not possible for the equilibrium characterization in the endogenous uncertainty case.

However, there is a characterization as the Variational Inequality (VI):

\[
Q(f) \cdot (f - f') \leq 0
\]

where \( f = \{ f_p \}_{p \in \mathcal{P}} \) - vector of path flows
\( Q(f) = \{ Q_p(f) \}_{p \in \mathcal{P}} \) - vector of path costs.

Why? At equilibrium, flow \( f \) is routed along minimum cost paths. Therefore, holding these path costs fixed as \( Q(f) \), any other flow \( f' \) would result in a higher overall cost
\[
Q(f) \cdot f' = \sum_{p \in \mathcal{P}} Q_p(f) \cdot f_p'
\]
Conversely, suppose a flow $f$ satisfies the VI \((***)\) for all $f'$. We will show that $f$ is in equilibrium. Suppose not. Then, there exists a flow-carrying path $p_1$ with cost greater than that of another path $p_2$ between the same source and destination: $Q_{p_1}(f) > Q_{p_2}(f)$.

Define $f'_p = f_p$ for all $p \neq p_1, p_2$,

$$f'_{p_1} = f_{p_1} - \varepsilon$$

$$f'_{p_2} = f_{p_2} + \varepsilon$$

Then,

$$\Delta(Q(f), f') = \sum_{p \in \mathcal{P}} Q_p(f) \cdot f'_p = \sum_{p \in \mathcal{P}} Q_p(f) \cdot f_p - \varepsilon \sum_{p \in \mathcal{P}} Q_p(f) + \varepsilon \sum_{p \in \mathcal{P}} Q_p(f)$$

$$< Q(f), f' \quad \text{contradiction.}$$

Therefore, the VI is an equivalent definition/characterization of the equilibrium flow $f$.

\textbf{Thm.} Equilibrium in the stochastic endogenous setting exists.

\textbf{Proof.} This follows from the theory of VI's, according to which a VI always has a solution if the cost operator $Q(f)$ is continuous and the feasible set is a compact convex set.

\textbf{Remark.} Interesting connection between VI and fixed points of functions/correspondences \((x = g(x))\). Typical approach for proving (mixed strategy) equilibrium existence is via fixed point theorems.

\textbf{Remark 2.} Uniqueness of equilibrium in this setting is open.

\textbf{Remark 3.} Thm will hold for any continuous path cost.
Succinct Representation of Equilibria

Both the convex program (**) and the VI (***) have exponentially many variables \( f_p \) for each path \( p \in \mathcal{P} \).

On the other hand, unlike deterministic settings, not every path flow decomposition of a given edge flow vector is at equilibrium:

\[
(a, 8) \quad (b, 1) \\
S \quad \infty \quad T \\
(a+1, 3) \quad (b-1, 8)
\]

\[
Q_a = a + b + 3 \\
Q_b = a + b + 3 \\
Q_{\infty} = a + b + 3 \\
Q_w = a + b + \sqrt{11} \to \text{higher cost.}
\]

(means, variance) pairs.

For given equilibrium edge flow vector sending \( \frac{1}{2} \) unit of flow on each edge, no flow can be sent on \( \infty \) path. By flow conservation, no flow can be sent on \( w \) path either, even though it is a minimum-cost path!

For given equilibrium edge flow, does there exist a succinct path flow decomposition (i.e., sending positive flow only on polynomially many paths)?

\[ (x_e)_{e \in E}, \quad \sum_{e \in E} d_e(x_e) dx + \lambda \sum_{p \in \mathcal{P}} \| f_p \|_2 \sqrt{\sum_{e \in E} d_e^2} \]

\[ \text{s.t.} \quad x_e = \sum_{p \in \mathcal{P}} f_p, \quad e \in E \]
\[ d_e = \sum_{p \in \mathcal{P}} f_p, \quad k = 1, \ldots, K \]
\[ f_p \geq 0. \]

Then: For equilibrium \( (x_e)_{e \in E} \), there exists a succinct flow decomposition \( (f_p)_{p \in \mathcal{P}} \) that uses at most \( 1|E| + |K| \) paths.

\[ \text{Pf: } \text{Equilibrium solves convex program (**): } \text{in exogenous setting} \]

For fixed \( x_e \), this is a linear program in \( f_p \) with \( 1|E| + |K| \) equality constraints \( \Rightarrow \) solution \( (f_p)_{p \in \mathcal{P}} \) in which number of nonzero variables is \( \leq 1|E| + |K| \). (bounded by equality constraint).
Price of Anarchy

Thm: In exogenous setting, the price of anarchy is \( \frac{4}{3} \), when the mean edge delays are linear: \( l_e(x) = a_e x + b_e \).

Prf: Denote by \( f \) the equilibrium flow and by \( f^* \) the socially optimal flow

\[
\text{minimizing } \sum_{p \in P} f^*_p Q_p(f^*) \quad \Rightarrow \quad C(f^*)
\]

Then, \( C(f) = \sum_{p \in P} f_p Q_p(f) \leq \sum_{p \in P} f^*_p Q_p(f) \quad \Rightarrow \quad \text{by the VI}
\]

\[
= \sum_{p \in P} f^*_p \left( \sum_{e \in p} l_e(f_e) + f^*_e \sigma_p \right) \quad \text{where } \sigma_p := \sqrt{\sum_{e \in p} \sigma_e^2}
\]

\[
= \sum_{e \in E} f^*_e l_e(f_e) + \frac{1}{4} \sum_{p \in P} f^*_p \sigma_p
\]

\[
\leq \sum_{e \in E} f^*_e l_e(f_e) + \frac{1}{4} \sum_{e \in E} a_e f_e^2 + \frac{1}{4} \sum_{p \in P} f^*_p \sigma_p
\]

\[
= \sum_{p \in P} f^*_p \left( \sum_{e \in p} l_e(f_e^*) + f^*_e b_e \right) + \frac{1}{4} \sum_{e \in E} a_e f_e^2 + \frac{1}{4} \sum_{p \in P} f^*_p \sigma_p
\]

\[
= \sum_{p \in P} f^*_p Q_p(f^*) + \frac{1}{4} \left( \sum_{e \in E} a_e f_e^2 + b_e f_e \right) + \frac{1}{4} \sum_{p \in P} f^*_p \sigma_p
\]

\[
= C(f^*) + \frac{1}{4} C(f)
\]

\[
\Rightarrow \quad \frac{3}{4} C(f) \leq C(f^*)
\]

\[
\Rightarrow \quad \frac{C(f)}{C(f^*)} \leq \frac{4}{3}
\]

QED