Mechanism Design continued
- Revelation Principle
- Optimal Mechanisms

So far, we saw examples of Mechanism Design in Auctions: First-Price Auctions and Second-Price Auctions (FPA, SPA respectively).

Of course, an auction is just one of many ways to sell an object to potential buyers.

What is the "best" way to allocate the object?

General model of a mechanism

- Seller has a single indivisible object for sale
- Set of potential buyers $N = \{1, 2, \ldots, n\}$.
- Buyers have private values $X_i$ drawn i.i.d.
  from cdf $F_i$ with pdf $f_i$, support $X_i = [0, w_i]$

- Assume value of object to seller is 0.
- Denote $\mathcal{X} = \prod_{i=1}^n X_i$ - the product set of buyers' values, and $X_{-i} = \prod_{j \neq i} X_j$

- $f(x) = f_1(x_1) \ldots f_n(x_n)$ is the joint density of
  (independent) values $x = (x_1, \ldots, x_n)$.

$$f_{-i}(x_{-i}) = \frac{f(x)}{f_i(x_i)}$$
def: A selling mechanism \((B, \pi, m)\) consists of:

1) A set of messages/bids/strategies \(B_i\) for \(i \in B\).

2) An allocation rule \(\pi : B \rightarrow \Delta\) where \(\Delta\) is the set of probability distributions over the set of buyers \(N\).

3) A payment rule \(m : B \rightarrow \mathbb{R}^N\).

An allocation rule specifies, as a function of bids \(b = (b_1, \ldots, b_m)\), the probability \(\pi_i(b)\) that buyer \(i\) will get the object.

A payment rule specifies the payment \(m_i(b)\) that buyer \(i\) must make.

Every mechanism defines a game of incomplete information among the buyers, where
- Strategies are \(\beta_i : [0, w_i] \rightarrow B_i\).
- Payoffs are the expected payoffs for a given strategy profile and selling mechanism.

**EQUILIBRIUM:** A strategy profile \(\beta(\cdot)\) is a Bayesian Nash Equilibrium of a mechanism if for all \(i\) and for all \(x_i\), given the strategies \(\beta_{-i}\) of other buyers, \(\beta_i(x_i)\) maximizes buyer \(i\)'s expected payoff.
Direct Mechanisms

A general mechanism could be very complicated without any assumptions on the (type) of messages $B_i$.

A special class of mechanisms, called direct mechanisms, are those for which the set of messages is the same as the set of buyer types or values: $B_i = \chi_i$ for all buyers $i$.

These mechanisms are called direct because every buyer is asked directly to report a value.

**def:** [DIRECT MECHANISM] A direct mechanism $(Q, M)$ consists of:

1. A function $Q: \chi \rightarrow \Delta$ where $Q_i(x)$ is the probability that buyer $i$ will get the object.

2. A function $M: \chi \rightarrow \mathbb{R}^n$, where $M_i(x)$ is the payment of buyer $i$.

If it is a Bayesian Nash Equilibrium for each buyer to report their type (value) $x_i$ correctly, we say that the direct mechanism has a truthful equilibrium.

The pair $(Q(x), M(x))$ is the outcome of the mechanism.
Revelation Principle

The following result allows us to restrict attention to direct mechanisms. It shows that outcomes resulting from any equilibrium of any mechanism can be replicated by a truthful equilibrium of some direct mechanism.

Proposition [REVELATION PRINCIPLE]:
Given a mechanism \((B, \pi, \mu)\) and an equilibrium \(\beta\) of that mechanism, there exists a direct mechanism \((Q, M)\), in which:

(i) it is a Bayesian Nash equilibrium for each buyer to report his value truthfully;

(ii) the outcomes are the same as in equilibrium \(\beta\) of the original mechanism.

Proof: Define the functions
\[ Q: X \rightarrow \Delta \text{ as } Q(x) = \pi(\beta(x)) \]
\[ M: X \rightarrow \mathbb{R}^n \text{ as } M(x) = \mu(\beta(x)). \]

Instead of buyers submitting message \(b_i = \beta(x_i)\), the mechanism asks the buyer to report their value and makes sure the outcome is the same as if buyers had submitted \(\beta(x_i)\).

In a sense, the direct mechanism does the "equilibrium calculation" for the buyers automatically.
Another way to think of direct mechanisms is as follows. Suppose that in mechanism $(B, \pi, \mu)$ each agent finds that when his type is $x_i$, choosing $\beta_i(x_i)$ is his best response to others' strategies.

If we have a mediator who says: "Tell me your type $x_i$ and I will play $\beta_i(x_i)$ for you," each agent will find truth telling to be an optimal strategy given that all other agents tell the truth.

So the direct mechanism is the mediator who does the equilibrium calculations $\beta_i(\cdot)$ for the agents.

We continue with the study of direct mechanisms. Since we saw that the significance of direct mechanisms is when truth-telling is an equilibrium, we formally define this property, called “Incentive Compatibility.”

For a given direct mechanism $(Q, M)$, define

$$q_i(z_i) = \int_{x_i} Q_i(z_i, x_i) f_i(x_i) \, dx_i$$

to be the probability that $i$ will get the object when he reports his value to be $z_i$ and all other buyers report their values truthfully, and

$$m_i(z_i) = \int_{x_i} M_i(z_i, x_i) f_i(x_i) \, dx_i$$

to be the expected payment of buyer $i$ when he reports $z_i$ and others report truthfully.
Incentive Compatibility (IC)

With the above definitions of \( q_i(z_i) \) and \( m_i(z_i) \), the expected payment of buyer \( i \) when his true value is \( x_i \) and he reports \( z_i \), assuming others tell the truth is

\[
q_i(z_i) \cdot x_i - m_i(z_i)
\]

prob. of winning  expected payment

\[
\text{definition: [ IC ] A direct revelation mechanism } (Q, M) \text{ is incentive compatible (IC) if}
\]

\[
q_i(x_i) \cdot x_i - m_i(x_i) \geq q_i(z_i) \cdot x_i - m_i(z_i) \quad \forall i, x_i, z_i.
\]

Define:

\[
U_i(x_i) = \max_{z_i \in X_i} \{ q_i(z_i) x_i - m_i(z_i) \}
\]

is the equilibrium payoff function.

\( U_i(x_i) \) is a maximum of affine functions, hence it is a convex function.

Incentive Compatibility is equivalent to:

\[
U_i(z_i) \geq U_i(x_i) + q_i(x_i) (z_i - x_i)
\]

for all \( z_i, x_i \), which follows from here:

\[
q_i(x_i) z_i - m_i(x_i) = q_i(x_i) x_i - m_i(x_i) + q_i(x_i) (z_i - x_i)
\]

\[
= U_i(x_i) + q_i(x_i) (z_i - x_i)
\]

So \( q_i(z_i) z_i - m_i(z_i) \geq q_i(x_i) z_i - m_i(x_i) \) - the IC condition above with \( x_i, z_i \) labels switched.
We write again the equivalent IC condition:

\[ U_i(z_i) = U_i(x_i) + q_i(x_i)(z_i - x_i) \]

This equation condition implies that for all \( x_i \),
\( q_i(x_i) \) is a subgradient of the function \( U_i \) at \( x_i \).

Hence, at every point that \( U_i \) is differentiable,

\[
\begin{align*}
U_i(x_i) & = q_i(x_i) \\
\Rightarrow \quad U_i(x_i) & = u_i(0) + \int_0^{x_i} q_i(t) \, dt. \quad (2)
\end{align*}
\]

Since \( U_i \) is convex, then \( q_i(t) \) is a nondecreasing function (recall, \( q_i(z_i) \) was the probability of agent winning when he reports \( z_i \) and others report truthfully).

Also, this shows that the expected payoff \( U_i(x_i) \) to a buyer in an IC direct mechanism \((Q, M)\) depends only on the allocation rule \( Q \).

From the above relations, we can also infer that incentive compatibility is equivalent to

(a) the function \( q_i \) being nondecreasing, and

(b) condition (2).
Revenue Equivalence

The payoff equivalence derived above leads to the general revenue equivalence principle: (since \( m_i(x_i) = u_i(x_i) + v_i(x_i) \); by defn of \( U_i \)).

Proposition [REVENUE EQUIVALENCE]:

If the direct mechanism \((Q, M)\) is incentive-compatible, then for all \(i\) and \(x_i\), the expected payment is given by

\[
m_i(x_i) = m_i(0) + q_i(x_i) \cdot x_i - \int_0^{x_i} q_i(t_i) \, dt_i.
\]

Thus, the expected payments in any two IC mechanisms with the same allocation rule are equivalent up to a constant.

Remarks: Corollary:

- Given two BNE of two different auctions, such that,
  for each agent \(i\):
  - for all \((x_1, \ldots, x_n)\), the probability of \(i\) getting the object is the same \(j\);
  - they have the same expected payment at value 0,

These equilibria will generate the same expected revenue for the seller.

This corollary generalizes the revenue equivalence result we stated in the context of auctions, namely revenue equivalence at the symmetric equilibrium of standard auctions (where the object goes to the highest bid).
Individual Rationality (IR) =
/ participation constraints /

- A seller cannot force a bidder to participate in an auction which offers him less expected utility than if he did not participate.

- If bidder does not participate in the auction, he will not get the object, and will not pay any money so his payoff would be zero.

**def:** [IR] We say that the direct mechanism \((\Omega, M)\) is individually rational \((IR)\) if for all agents \(i\) and values \(x_i\), the equilibrium expected payoff is nonnegative: \(U_i(x_i) \geq 0\).

If the mechanism is IC, then from Eq. (2) above
\[
(U_i(x_i) = U_i(0) + \int_0^{x_i} q_i(t_i) dt_i),
\]
\(\Box\) \(IR\) is equivalent to
\[
U_i(0) \geq 0;\]

Since \(U_i(0) = -m_i(0)\), IR is also equivalent to
\[
m_i(0) \leq 0.\]