Computing approximate Nash Equilibria

Based on "Playing Large Games using Simple Strategies" by Lipton, Markakis & Mehta. (ACM EC 2003)

This paper shows that for a finite 2-player game with bounded payoffs, an $\varepsilon$-NE can be computed in polynomial time. Denote the payoff matrices $(A, B)$.

Recall:

**Definition:** Given a scalar $\varepsilon > 0$, a mixed strategy profile $(x, y)$ in a 2-player game is an $\varepsilon$-Nash Equilibrium ($\varepsilon$-NE) if

$$x^T A y \geq \bar{x}^T A y - \varepsilon, \quad \text{for all } \bar{x} \in \mathcal{X}$$

$$x^T B y \geq x^T B \bar{y} - \varepsilon, \quad \text{for all } \bar{y} \in \mathcal{Y}.$$

Assume both players have $N \geq 2$ strategies.

The algorithm for computing approximate equilibria uses "simple mixed strategies", defined as follows:

**Definition [simple mixed strategy]:** A mixed strategy of player $i$ is called $k$-uniform if it is the uniform distribution on a multiset $S_i$ of the pure strategies $S_i$ with $|S_i| = k$.

For example, for a player with 3 pure strategies $(a, b, c)$, both $x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $x = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$ are 3-uniform strategies.

uniform on $(a, b, c)$ uniform on $(a, a, b)$. 
Theorem: Assume that all the entries of the matrices \( A, B \) are between 0 and 1. Let \((x^*, y^*)\) be a mixed NE and let \( \varepsilon > 0 \). For all \( k \geq \frac{12 \ln n}{\varepsilon^2} \), there exists a pair of \( k \)-uniform strategies \((x, y)\) such that:

(a) \((x, y)\) is an \( \varepsilon \)-NE;
(b) The two players get almost the same payoff as in the NE \((x^*, y^*)\):

\[
\left| x^T A y - x^* A y^* \right| < \frac{\varepsilon}{2}
\]
\[
\left| x^T B y - x^* A y^* \right| < \frac{\varepsilon}{2}
\]

The proof relies on the probabilistic sampling argument. We will give a high-level sketch of the proof.

Proof uses Hoeffding's inequality:

Lemma [Hoeffding's inequality]: Let \( X_1, X_2, \ldots, X_K \) be independent random variables with \( X_i \in [0, 1] \). Let \( X = \frac{1}{K} \sum_{i=1}^{K} X_i \). Then,

\[
\Pr \left[ \left| X - \mathbb{E}[X] \right| > t \right] \leq 2e^{-2kt^2}
\]

For the given \( \varepsilon > 0 \), fix \( k \geq \frac{12 \ln n}{\varepsilon^2} \). Form a multiset \( \overline{X} \) by sampling \( k \) times from the set of pure strategies of the row player, independently at random according to distribution \( x^* \). Similarly, form a multiset \( \overline{Y} \) according to distribution \( y^* \).

Let \( x^* \) be the mixed strategy for the row player which assigns prob. \( \frac{1}{k} \) to each member of \( \overline{X} \), and 0 to the other pure strategies. Similarly, define \( y^* \).
Denote by $e_i$ the $i$th pure strategy of the row player, and by $f_j$ the $j$th pure strategy of the column player.

To analyze the prob. that $x^*, y^*$ is an $\varepsilon$-NE, it suffices to consider only deviations to pure strategies.

Define events:

$$\Phi_1 = \{ |(x_i^T R) - (x_i^* R y^*)| < \frac{\varepsilon}{2} \}$$

$$\Pi_{i,i} = \{ x_i^T R y < x_i^T R y + \varepsilon \} \quad \text{for } i = 1, \ldots, n$$

$$\Phi_2 = \{ |x^T C y - x^* C y^*| < \frac{\varepsilon}{2} \}$$

$$\Pi_{2,j} = \{ x^T C e_j < x^T C y + \varepsilon \} \quad \text{for } j = 1, \ldots, n$$

$$\text{GOOD} = \Phi_1 \cap \bigcap_{i=1}^n \Pi_{i,i} \cap \bigcap_{j=1}^n \Pi_{2,j}$$

We wish to show that $\Pr(\text{GOOD}) > 0$: this would imply that there exist sets $\overline{x}, \overline{y}$ such that the corresponding $(x', y')$ satisfy the conditions of the theorem.

First, we bound $\Pr(\Phi_1), \Pr(\Phi_2)$. For $\Phi_1$, define

$$\Phi_{1a} = \{ |x^T R y^* - x^* T R y^*| < \frac{\varepsilon}{4} \} \quad (1)$$

$$\Phi_{1b} = \{ |x^T R y - x^T R y^*| < \frac{\varepsilon}{4} \} \quad (2)$$

Then $\Phi_{1a} \cap \Phi_{1b} \subseteq \Phi_1$, since given (1) & (2),

by a inequality, $|x^T R y - x^* T R y^*| \leq |x^T R y - x^T R y^*| + |x^* T R y^* - x^* T R y^*| < \frac{\varepsilon}{2}$. 

\[
\begin{align*}
\text{GOOD} &= \Phi_1 \cap \bigcap_{i=1}^n \Pi_{i,i} \cap \bigcap_{j=1}^n \Pi_{2,j} \\
\text{GOOD} &= \Phi_1 \cap \bigcap_{i=1}^n \Pi_{i,i} \cap \bigcap_{j=1}^n \Pi_{2,j} \\
\end{align*}
\]
\[ x^T R y^* \text{ is a sum of } k \text{ indep. random variables in } [0, 1] \text{ each with expected value } x^T R y^*, \]

since
\[
x^T R y^* = \sum_{i=1}^{n} x_i (Ry^*)_i
\]
\[
= \sum_{j=1}^{K} (Ry^*)_j
\]

where \( j \in \{0, 2, \ldots, n\} \) is the index of the \( j \text{th pure strategy in the multiset } X \).

\[ \Rightarrow \text{ By Hoeffding's inequality,} \]
\[ \Pr \left( \Phi_1^c \right) \leq 2 e^{-\frac{K \varepsilon^2}{8}}, \text{ where } \Phi_1^c \text{ is the complement of } \Phi. \]

Similarly,
\[ \Pr \left( \Phi_2^c \right) \leq 2 e^{-\frac{K \varepsilon^2}{8}}, \]

\[ \Rightarrow \Pr \left( \Phi_1^c \right) \leq 4 e^{-\frac{K \varepsilon^2}{8}} \text{ by the union bound.} \]

Similarly,
\[ \Pr \left( \Phi_2^c \right) \leq 4 e^{-\frac{K \varepsilon^2}{8}}. \]

For bounding the probabilities of events \( \Pi_{1,i} \cap \Pi_{2,i} \): Define auxiliary events:

\[ \Psi_{1,i} = \{ x_i^T R y < e_i R y^* + \frac{\varepsilon}{2} \} \]

Observe that \( \Psi_{1,i} \cap \Phi_1 \subseteq \Pi_{1,i} \) because given \( \Psi_{1,i}, \Phi_1 \), we have
(Ψ_i) \[ e_i R y < e_i R y^* + \frac{\varepsilon}{2} \quad \text{and} \]

(Φ_i) \[ |x^T R y - x^* R y^*| \leq \frac{\varepsilon}{2} \]

\[ \Rightarrow \quad -\frac{\varepsilon}{2} < x^T R y - x^* R y^* < \frac{\varepsilon}{2} \]