Optimal Mechanisms

- An optimal mechanism is defined as one that maximizes the expected revenue among all mechanisms that are IR and IC.
- Goal here is to design this mechanism.
- WLOG (without loss of generality), we focus on direct revelation mechanisms.

Consider the direct mechanism \((Q, m)\). The expected revenue to the seller is

\[
E[R] = \sum_{i \in N} E[m_r(x_i)],
\]

where

\[
E[m_r(x_i)] = \int_0^{w_i} m_i(x_i) f_i(x_i) \, dx_i
\]

\[
= m_i(0) + \int_0^{w_i} q_i(x_i) x_i f_i(x_i) \, dx_i
\]

\[
- \int_0^{w_i} \int_0^{x_i} q_i(t_i) dt_i f_i(x_i) \, dx_i \tag{*}
\]

(from the Revenue Equivalence proposition where

\[
m_i(x_i) = m_i(0) + q_i(x_i) x_i - \int_0^{x_i} q_i(t_i) dt_i.
\]

Changing the order of integration in the third term gives:

\[
E[m_r(x_i)] = m_i(0) + \int_0^{w_i} \left( x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) q_i(x_i) f_i(x_i) \, dx_i
\]

\[
= m_i(0) + \int_\chi \left( x - \frac{1 - F_i(x)}{f_i(x)} \right) Q_i(x) f(x) \, dx,
\]

using the definition of \(\chi\) as earlier (page 5).
Changing the order of integration in the last term gives:

$$\int_0^{x_i} \int_0^{t_i} q_i(t_i) \, dt_i \, f(x_i) \, dx_i = \int_0^{w_i} \int_{t_i}^{w_i} q_i(t_i) f_i(x_i) \, dx_i \, dt_i$$

$$0 \leq t_i \leq x_i \iff t_i \leq x_i \leq w_i \quad = \int_0^{w_i} \left(1 - F_i(t_i)\right) q_i(t_i) \, dt_i.$$

Therefore, (*) can be rewritten as:

$$E[m_i(X_i)] = m_i(0) + \int_0^{w_i} q_i(x_i) x_i f_i(x_i) \, dx_i$$

$$- \int_0^{w_i} \left(1 - F_i(x_i)\right) q_i(x_i) \, dx_i$$

$$\leq \text{relate to } t_i \text{ above}$$

$$= m_i(0) + \int_0^{w_i} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}\right) q_i(x_i) f_i(x_i) \, dx_i$$

$$= m_i(0) + \int_{x_i}^{w_i} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}\right) Q_i(x) f(x) \, dx,$$

using the definition of $q_i(x_i)$ from earlier (page 5).

The Optimal Mechanism Design Problem is:

$$\text{maximize } E[R]$$

subject to

- IC
- IR

($\rightarrow q_i$ nondecreasing and (2) $(\rightarrow m_i(0) \leq 0$).
Optimal Mechanism continued

def: The virtual valuation of a buyer with value $x_i$ is

$$\Psi_i(x_i) = x_i - \frac{1-F_i(x_i)}{f_i(x_i)}$$

We say that the design problem is regular when the virtual valuation $\Psi_i(x_i)$ is strictly increasing in $x_i$. Under this regularity assumption we can neglect the IR and IC constraints WLOG.

From the computation of expected revenue & payments above, the seller wishes to find $Q, m$ to maximize:

$$\sum_{i \in N} m_i(0) + \int X \left( \sum_{i \in N} \Psi_i(x_i) Q_i(x) \right) f(x) \, dx$$

The following is an optimal mechanism:

**Allocation Rule:** $Q_i(x) > 0 \iff \Psi_i(x_i) = \max_{j \in N} \Psi_j(x_j) > 0$.

**Payment Rule:** $M_i(x) = Q_i(x) \cdot x_i - \int_0^{x_i} Q_i(\xi, x_i-x_i) \, d\xi$

This mechanism satisfies IR & IC because:

1) [IR] $M_i(0, x_i) = 0 \forall x_i \implies m_i(0) = 0 \implies IR$.

2) [IC] $\Psi_i(\xi_i) < \Psi_i(x_i)$ for any $\xi_i < x_i$ by the regularity assumption.

$$\implies Q_i(\xi_i, x_i) \leq Q_i(x_i, x_i-x_i) \forall x_i$$

$$\implies q_i(\xi_i) \leq q_i(x_i) \implies q_i \text{ is nondecreasing}.$$ From this and the payment rule, IC is satisfied.
Optimal Mechanism: Revenue

The optimal revenue is the expectation of the highest virtual valuation, provided it is nonnegative:

$$E[ \max \{ \Psi_i(x_i), ..., \Psi_n(x_n) \} ] = 0$$

Define $y_i(x_{-i}) = \inf \{ z_i \mid \Psi_i(z_i) \geq 0, \Psi_i(z_i) \geq \Psi_j(x_j), \forall j \neq i \}$ as the smallest value for $i$ that wins against $x_{-i}$.

Then, we can re-write:

$$Q_i(z_i, x_{-i}) = \begin{cases} 1 & \text{if } z_i > y_i(x_{-i}) \\ 0 & \text{if } z_i < y_i(x_{-i}) \end{cases}$$

Then,

$$\int_0^{x_i} Q_i(z_i, x_{-i}) \, dz_i = \begin{cases} x_i - y_i(x_{-i}) & \text{if } x_i > y_i(x_{-i}) \\ 0 & \text{if } x_i < y_i(x_{-i}) \end{cases}$$

$$\Rightarrow M_i(x) = \begin{cases} y_i(x_{-i}) & \text{if } Q_i(x) = 1 \\ 0 & \text{if } Q_i(x) = 0 \end{cases}$$

So: only the winning buyer pays, and he pays the smallest value that would make him win.

**Symmetric Case:** If distributions of values are identical across buyers ($f_i = f$), so that $\Psi_i = \Psi = \Psi_i$, then

$$y_i(x_{-i}) = \max \{ \Psi^{-1}(0), \max_{j \neq i} x_j \}$$

**Proposition:** If the design problem is regular and symmetric, a second-price auction with reservation price $r^* = \Psi^{-1}(0)$ is an optimal mechanism.

**Note:** Unlike first & second price auction, the optimal mechanism is not efficient i.e. winner may not have highest value