Exhaustive Proofs

• Prove for every element in the domain
• Ex: \((n + 1)^3 \geq 3^n\) for \(n \in \{0, 1, 2, 3, 4\}\)
  – Exhaustive Proof:
  – \((0 + 1)^3 \geq 3^0\)
    • \(1 \geq 1\)
  – \((1 + 1)^3 \geq 3^1\)
    • \(8 \geq 3\)
  – \((2 + 1)^3 \geq 3^2\)
    • \(27 \geq 9\)
  – \((3 + 1)^3 \geq 3^3\)
    • \(64 \geq 27\)
  – \((4 + 1)^3 \geq 3^4\)
    • \(125 \geq 81\)
  – □
Proof by Cases

• Prove for every case in the theorem
• Ex: if $n$ is an integer, then $n^2 \geq n$
  – Proof by cases:
  – Case $n \geq 1$:
    • $n \cdot n \geq 1 \cdot n$
    • $n^2 \geq n$
  – Case $n \leq 1$:
    • $n^2 \geq n$, since $n^2$ is positive and $n$ is negative
  – Case $n = 0$:
    • $0^2 \geq 0$
  – The claim holds in all cases □
Leveraging Proof by Cases

• When you can’t consider every case all at once
• When there’s no obvious way to start, but extra information in each case helps
• Ex: Formulate and prove a conjecture about the final digit of perfect squares
  – List some perfect squares
  – Look at the final digit \((13^2 = 169)\)
  – See a pattern?
Theorem: The final digit of a perfect square is 0, 1, 4, 5, 6, or 9.

Proof:

- $n = 10a + b$, $b \in \{0, 1, 2, ..., 9\}$
- $n^2 = (10a + b)^2 = 10(10a^2 + 2ab) + b^2$
  
  - $n^2$ and $b^2$ have the same final digit
- Case $b = 0$:
  - $0^2 = 0$, $n^2$ ends in 0
- Case $b \in \{1, 9\}$:
  - $1^2 = 1, 9^2 = 81$, $n^2$ ends in 1
- Case $b \in \{2, 8\}$:
  - $2^2 = 4, 8^2 = 64$, $n^2$ ends in 4
- Case $b \in \{3, 7\}$:
  - $3^2 = 9, 7^2 = 49$, $n^2$ ends in 9
- Case $b \in \{4, 6\}$:
  - $4^2 = 16, 6^2 = 36$, $n^2$ ends in 6
- Case $b = 5$:
  - $5^2 = 25$, $n^2$ ends in 5

$\square$
Leveraging Proof by Cases

• Theorem: \( x^2 + 3y^2 = 8 \) has no integer solutions
• Proof by Cases:
  – \( x^2 \leq 8 \) \( |x| < 3 \)
  – \( 3y^2 \leq 8 \) \( |y| < 2 \)
  – 15 cases
    • \( x \in \{-2, -1, 0, 1, 2\} \)
    • \( y \in \{-1, 0, 1\} \)
  – 6 cases
    • \( x^2 \in \{0, 1, 4\} \)
    • \( y^2 \in \{0, 1\} \)
  – \leq 6 cases
    • \( x^2 \in \{0, 1, 4\} \)
    • \( 3y^2 \in \{0, 3\} \)
    • \( 4 + 3 = 7 \neq 8 \)
  – \( \therefore \) no integer solutions \( \square \)
Without Loss of Generality (wlog)

- Assert that the proof for one case can be reapplied with only straightforward changes to prove other specified cases.
- Ex: Let \( x, y \) be integers. If \( xy \) and \( x + y \) are both even, then \( x \) and \( y \) are both even.
  - Proof
  - Use contraposition
    - \( (x \text{ is odd}) \lor (y \text{ is odd}) \rightarrow ((xy \text{ odd}) \lor (x + y \text{ is odd})) \)
  - Assume \( (x \text{ is odd}) \lor (y \text{ is odd}) \)
  - Wlog, assume \( x \) is odd.
  - Case \( y \) even:
    - \( x + y = (\text{odd}) + (\text{even}) = \text{odd} \checkmark \)
  - Case \( y \) odd:
    - \( xy = (\text{odd})(\text{odd}) = \text{odd} \checkmark \)
  - \( \therefore (x \text{ is odd}) \lor (y \text{ is odd}) \rightarrow ((xy \text{ odd}) \lor (x + y \text{ is odd})) \)
  - It follows that \( ((xy \text{ even}) \land (x + y \text{ is even})) \rightarrow ((x \text{ even}) \land (y \text{ is even})) \)
  \( \square \)
• Exhaustive proof and proof by cases are only valid when you prove **every** case

• Ex: every positive integer is the sum of 18 fourth powers of integers.
  – 79 is the counterexample

• Ex: if $x$ is a real number, then $x^2$ is positive
  – Case $x$ is positive:
    • $(\text{positive})(\text{positive}) = (\text{positive})$ ✓
  – Case $x$ is negative:
    • $(\text{negative})(\text{negative}) = (\text{positive})$ ✓
  – Case $x$ is zero:
    • $(\text{zero})(\text{zero}) = \text{zero}$ X
Existence Proofs

• To prove \( \exists x P(x) \)
  – Constructive: find a witness
  – Nonconstructive: shown without witness

• Ex: Show that there exists some integer that is expressible as the sum of 2 cubes in 2 different ways
  – Constructive proof: \( 1729 = 10^3 + 9^3 = 12^3 + 1^3 \)
Existence Proofs

• Ex: Show that there exists 2 irrational numbers \( x, y \) such that \( x^y \) is rational
  – Nonconstructive proof:
    • Known: \( \sqrt{2} \) is irrational
    • Consider \( x = y = \sqrt{2} \), so \( x^y = \sqrt{2}^{\sqrt{2}} \)
    • If \( \sqrt{2}^{\sqrt{2}} \) is rational, then \( x = y = \sqrt{2} \) are the witnesses
    • If \( \sqrt{2}^{\sqrt{2}} \) is irrational, then let \( x = \sqrt{2}^{\sqrt{2}} \) and \( y = \sqrt{2} \)
      \[ x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2 \]
      – \( x = \sqrt{2}^{\sqrt{2}} \) and \( y = \sqrt{2} \) are the witnesses
    • Either way, we found irrational \( x, y \) such that \( x^y \) is rational
      – We just don’t know which value to use for \( x \)
Uniqueness Proofs

• “exactly one element satisfies $P(x)$”
  – Existence: $\exists x \ P(x)$
  – Uniqueness: $\forall y \ P(y) \rightarrow (y = x)$
  – $\exists x \forall y \ P(x) \land (P(y) \rightarrow (y = x))$
  – $\exists x \forall y \ P(y) \leftrightarrow (y = x)$
Uniqueness Proofs

• $ar + b = 0$ has a unique solution when $a, b$ are real and $a \neq 0$
  – Proof:
    – Existence
      • $ar + b = 0$
      • $ar = -b$
      • $r = \frac{-b}{a}$
    – Uniqueness
      • Assume $\exists s \ as + b = 0$
      • Then, $ar + b = as + b$
      • $ar + b - b = as + b - b$
      • $\frac{ar}{a} = \frac{as}{a}$
      • $r = s$
    – $\therefore r = \frac{-b}{a}$ is the solution to $ar + b = 0$
Proof Strategies

• Forward
  – Start with premises, plug and chug to the conclusion.
    • Direct proof
  – Start with negation of conclusion, plug and chug to negation of premises
    • Indirect proof

• Backward
  – Work backwards from the conclusion to find the correct steps for a direct proof
Backward Reasoning

• Prove that $\frac{(x+y)}{2} \geq \sqrt{xy}$ for positive reals $x, y$

\[ - \frac{(x+y)}{2} \geq \sqrt{xy} \]

\[ - \frac{(x+y)^2}{4} \geq xy \]

\[ - (x + y)^2 \geq 4xy \]

\[ - x^2 + 2xy + y^2 \geq 4xy \]

\[ - x^2 - 2xy + y^2 \geq 0 \]

\[ - (x - y)^2 \geq 0 \]

\[ \therefore \frac{(x+y)}{2} \geq \sqrt{xy} \]
Adapting Existing Proofs

• Theorem: $\sqrt{3}$ is irrational
  – Proof: $\sqrt{2}$ is irrational, use that proof

• $\sqrt{n}$ is irrational when $n$ is not a perfect square.
  – Same kind of proof: contradiction
Tilings

• Can a standard checkerboard (8 × 8) be tilled by dominoes?
  – Yes.

• Can a standard checkerboard with one corner missing be tilled by dominoes?
  – No.
  – 63 squares is not an even number of squares

• Can a standard checkerboard with two opposite corners missing be tilled by dominoes?
  – No.
  – Opposite corners have the same color.
    • 32 black + 30 white cannot be tiled because each domino covers 1 white and 1 black square