Global Knot Insertion Algorithms

Scott Schaefer
Department of Computer Science
Texas A&M University

Ron Goldman
Department of Computer Science
Rice University
Part I:
Introduction to Knot Insertion
Knot Insertion Algorithm

Input

- $T = \{t_1, \ldots, t_{\nu+n}\}$ -- knot sequence
- $P = \{P_0, \ldots, P_{\nu}\}$ -- control points
- $\Gamma = \{\tau_1, \ldots, \tau_{\mu+n}\}$ -- new knot sequence -- $\Gamma \supset T$

Output

- $Q = \{Q_0, \ldots, Q_{\mu}\}$ -- new control points

Constraint

- $\sum_{k=0}^{\nu} N_{k,n}(t | T)P_k = \sum_{k=0}^{\mu} N_{k,n}(\tau | \Gamma)Q_k$
B-spline Curve

\[ S(\tau) \]
Knot Insertion

\[ Q_0 = P_0 \]

\[ Q_1 \]

\[ Q_2 \]

\[ Q_3 \]

\[ Q_4 = P_3 \]

\[ S(\tau) \]
Theorems

Existence

• All splines are B-splines.

Convergence

• The control polygons generated by knot insertion converge to the B-spline curve for the original control polygon as the knot spacing approaches zero.

Corner Cutting

• Knot insertion is a corner cutting procedure.
Applications of Knot Insertion

- Rendering
- Intersection
- Conversion from B-spline to Piecewise Bezier Form
- Proof of the Variation Diminishing Property
Types of Knot Insertion Algorithms

Local Knot Insertion

- $T = \{t_1, \ldots, t_{v+n}\}$
- $\Gamma = \{t_1, \ldots, t_k, u_{k,1}, \ldots, u_{k,d_k}, t_{k+1}, \ldots, t_{v+n}\}$

Global Knot Insertion

- $T = \{t_1, \ldots, t_{v+n}\}$
- $\Gamma = \{t_1, u_{1,1}, \ldots, u_{1,d_1}, t_2, \ldots, t_{v+n-1}, u_{v+n-1,1}, \ldots, u_{v+n-1,d_{v+n-1}}, t_{v+n}\}$
### Examples of Knot Insertion Algorithms

#### Local Knot Insertion Algorithms
- Boehm’s Algorithm
- Oslo Algorithm
- Sablonniere’s Algorithm
- Factored Knot Insertion

#### Global Knot Insertion Algorithms
- Chaikin’s Algorithm
- Lane-Riesenfeld Algorithm
- Goldman-Warren Algorithm
- Schaefer’s Algorithm -- NEW
Myths

Global Knot Insertion

• Works Only for Uniform Knot Sequences
  -- Knots in Arithmetic Progression -- Lane-Riesenfeld (1973)
    \[ t_{k+1} = t_k + \alpha \]
  -- Knots in Geometric or Affine Progression -- Goldman-Warren (1993)
    \[ t_{k+1} = \beta t_k + \alpha \]
• Does not Apply to Arbitrary Knot Sequences

Blossoming

• Provides Only New Proofs of Already Known Results
  -- No New Results
  -- No New Insights
Part II:
Local Knot Insertion Algorithms
Knot Insertion Algorithms from Blossoming

- Boehm’s Algorithm -- Inserts one new knot at a time

- Oslo Algorithm -- Computes one new control point at a time

- Sablonniere’s Algorithm -- Local change of basis algorithm

- Factored Knot Insertion -- Forward differencing for knot insertion
Blossoming

Symmetry
- \( p(u_1,\ldots,u_n) = p(u_{\sigma(1)},\ldots,u_{\sigma(n)}) \) for any permutation \( \sigma \) of \( \{1,\ldots,n\} \)

Multiaffine
- \( p(u_1,\ldots,(1-\alpha)u_k + \alpha w_k,\ldots,u_n) = (1-\alpha)p(u_1,\ldots,u_k,\ldots,u_n) + \alpha p(u_1,\ldots,w_k,\ldots,u_n) \)

Diagonal
- \( p(t,\ldots,t) = P(t) \)

Dual Functional Property
- \( P(t) = \sum_k N_{k,n}(t)P_k \Rightarrow P_k = p(t_{k+1},\ldots,t_{k+n}) \) (B-spline Curves)
- \( P(t) = \sum_k B_k^n(t)P_k \Rightarrow P_k = p(a,\ldots,a,b,\ldots,b) \) (Bezier Curves)
Properties of the Blossom

Existence and Uniqueness

• Every Degree $n$ Polynomial $P(t)$ has a Unique Blossom $p(u_1,\ldots,u_n)$

Power Law

• Each Parameter $u_1,\ldots,u_n$ Appears to at Most the First Power
• Equivalent to Multiaffine Axiom
Examples (Existence)

Monomials

\[ P(t) = 1 \implies p(u_1,u_2,u_3) = 1 \]

\[ P(t) = t \implies p(u_1,u_2,u_3) = \frac{u_1 + u_2 + u_3}{3} \]

\[ P(t) = t^2 \implies p(u_1,u_2,u_3) = \frac{u_1u_2 + u_2u_3 + u_3u_1}{3} \]

\[ P(t) = t^3 \implies p(u_1,u_2,u_3) = u_1u_2u_3 \]

Cubics

\[ P(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0 \]

\[ p(u_1,u_2,u_3) = a_3 u_1 u_2 u_3 + a_2 \frac{u_1u_2 + u_2u_3 + u_3u_1}{3} + a_1 \frac{u_1 + u_2 + u_3}{3} + a_0 \]
Blossoming Diagrams -- Multiaffine Property

\[ p(t_2, t_3, u) = \frac{t_4 - u}{t_4 - t_1} p(t_1, t_2, t_3) + \frac{u - t_1}{t_4 - t_1} p(t_2, t_3, t_4) \]

\[ u = \frac{t_4 - u}{t_4 - t_1} t_1 + \frac{u - t_1}{t_4 - t_1} t_4 \quad \Rightarrow \quad p(t_2, t_3, u) = \frac{t_4 - u}{t_4 - t_1} p(t_1, t_2 t_3) + \frac{u - t_1}{t_4 - t_1} p(t_2, t_3 t_4) \]

\[ p(t_2, t_3, u) = \frac{t_4 - u}{t_4 - t_1} p(t_1, t_2 t_3) + \frac{u - t_1}{t_4 - t_1} p(t_2, t_3 t_4) \quad \Leftrightarrow \quad t_2 t_3 u = \frac{t_4 - u}{t_4 - t_1} t_1 t_2 t_3 + \frac{u - t_1}{t_4 - t_1} t_2 t_3 t_4 \]
Blossoming Diagrams -- Multiaffine Property

\[
\begin{align*}
&u = \frac{t_{n+1} - u}{t_{n+1} - t_1} t_1 + \frac{u - t_1}{t_{n+1} - t_1} t_{n+1} \\
p(t_2, \ldots, t_n, u) &= \frac{t_{n+1} - u}{t_{n+1} - t_1} p(t_1, \ldots, t_n) + \frac{u - t_1}{t_{n+1} - t_1} p(t_2, \ldots, t_{n+1}) \\
t_2 \cdots t_n u &= \frac{t_{n+1} - u}{t_{n+1} - t_1} t_1 \cdots t_n + \frac{u - t_1}{t_{n+1} - t_1} t_2 \cdots t_{n+1}
\end{align*}
\]
Boehm’s Algorithm

New Control Points

Original Control Points

Original Knot Sequence: \( \ldots, t_1, t_2, t_3, t_4, t_5, t_6, \ldots \)

New Knot Sequence: \( \ldots, t_1, t_2, t_3, u, t_4, t_5, t_6, \ldots \)
Oslo Algorithm

New Control Point

Original Control Points

Original Knot Sequence: \( \ldots, t_1, t_2, t_3, t_4, t_5, t_6, \ldots \)

New Knot Sequence: \( \ldots, t_1, t_2, t_3, u_1, \ldots, u_d, t_4, t_5, t_6, \ldots \)
Part III:
Global Knot Insertion Algorithms
Chaikin’s Algorithm:
Quadratic B-splines -- Uniform Knots

\[ \frac{3P_0 + P_1}{4} \quad \frac{P_0 + 3P_1}{4} \quad \frac{3P_1 + P_2}{4} \quad \frac{P_1 + 3P_2}{4} \quad \frac{3P_2 + P_3}{4} \quad \frac{P_2 + 3P_3}{4} \ldots \]

\[ P_0 \quad \frac{P_0 + P_1}{2} \quad P_1 \quad \frac{P_1 + P_2}{2} \quad P_2 \quad \frac{P_2 + P_3}{2} \quad P_3 \ldots \]

Split and Average
Lane-Riesenfeld Algorithm:

Cubic B-splines -- Uniform Knots

\[
\begin{align*}
\frac{4P_0 + 4P_1}{8} & \quad \frac{P_0 + 6P_1 + P_2}{8} & \quad \frac{4P_1 + 4P_2}{8} & \quad \frac{P_1 + 6P_2 + P_3}{8} & \quad \frac{4P_2 + 4P_3}{8} & \ldots \\
\frac{3P_0 + P_1}{4} & \quad \frac{P_0 + 3P_1}{4} & \quad \frac{3P_1 + P_2}{4} & \quad \frac{P_1 + 3P_2}{4} & \quad \frac{3P_2 + P_3}{4} & \quad \frac{P_2 + 3P_3}{4} & \ldots \\
P_0 & \quad \frac{P_0 + P_1}{2} & \quad \frac{P_1 + P_2}{2} & \quad \frac{P_2 + P_3}{2} & \quad \frac{P_3}{2} & \quad \ldots \\
P_0 & \quad P_0 & \quad P_1 & \quad P_1 & \quad P_2 & \quad P_2 & \quad P_3 & \quad P_3 & \ldots \\
\end{align*}
\]

Split and Average
Proofs of Lane-Riesenfeld Algorithm

1. Continuous Convolution of B-spline Basis Functions

2. De Boor Recurrence (Induction)

2'. De Boor Recurrence (Induction) ⇒ Oslo Algorithm
New Proof of Lane-Riesenfeld Algorithm

3. Blossoming

Lane-Riesenfeld Algorithm -- Blossoming Interpretation

Quadratic B-splines

Uniform Knots
Lane-Riesenfeld Algorithm

*Quadratic B-splines*

1. Double the Control Points
2. Convert to Piecewise Bezier Form
3. Insert New Knots Using Boehm’s Knot Insertion Algorithm
Lane-Riesenfeld Algorithm -- Quadratic B-splines

Arbitrary Knots

\[ u = \frac{t_{k+1} - u}{t_{k+1} - t_k} t_k + \frac{u - t_k}{t_{k+1} - t_k} t_{k+1} \]
Schaefer’s Algorithm -- Quadratic B-splines

Arbitrary Knots

\[ u = \frac{t_{k+1} - u}{t_{k+1} - t_k} t_k + \frac{u - t_k}{t_{k+1} - t_k} t_{k+1} \]
**Quadratic B-splines**

*Schaefer’s Algorithm*

- Inserts New Knots in First Round
- Not Necessary to Convert to Piecewise Bezier Form
- Faster than Lane-Riesenfeld -- Half the Work in the Second Round
Knot Insertion: Quadratic B-splines

Lane – Riesenfeld: $\bullet = t_k t_{k+1}$

Schaefer: $\bullet = u_k t_{k+1}$
Lane-Riesenfeld Algorithm -- Cubic B-splines

Uniform Knots
Lane-Riesenfeld Algorithm -- Cubic B-splines

\[
\begin{align*}
(\frac{9}{4}, 1, 2) &= \frac{1}{2}(1, 1\frac{1}{2}, 2) + \frac{1}{2}(1, 2, 3) \\
(\frac{7}{4}, 2, 3) &= \frac{1}{2}(1, 2, 3) + \frac{1}{2}(2, 2\frac{1}{2}, 3)
\end{align*}
\]

\[
? = \frac{1}{4}(1, 1\frac{1}{2}, 2) + \frac{2}{4}(1, 2, 3) + \frac{1}{4}(2, 2\frac{1}{2}, 3)
\Rightarrow \quad ? = \frac{1}{4}(1, 1\frac{1}{2}, 2) + \frac{3}{4}\left(\frac{2}{3}(1, 2, 3) + \frac{1}{3}(2, 2\frac{1}{2}, 3)\right)
\]

\[
? = \frac{1}{4}(1, 1\frac{1}{2}, 2) + \frac{3}{4}(1\frac{1}{2}, 2, 3) = \left(1\frac{1}{2}, 2, 2\frac{1}{2}\right)
\]
Lane-Riesenfeld Algorithm -- Quartic B-splines

Uniform Knots
Lane-Riesenfeld Algorithm

- Build Algorithm for Next Degree Atop Algorithm for Previous Degree
- Append One Additional Round of Averaging
- Harder and Harder to Prove by Blossoming
Schaefer’s Algorithm -- Cubic B-splines

\[ u = \frac{t_k - u}{t_k - t_j}t_j + \frac{u - t_j}{t_k - t_j}t_k \]
Schaefer’s Algorithm

Cubic B-splines

• Build Atop Algorithm for Quadratic B-splines

• Append Next Original Knot to Each of the Blossoms on the First Two Stages
  -- Example: $u_0 t_1 \rightarrow u_0 t_1 t_2$

• Promote Every Other Point to the Next Stage with No Additional Computation

• Introduce New Knots Using the Multiaffine Property of the Blossom

• Easy to Prove by Blossoming
Alternative Algorithm -- Cubic B-splines

\[ u = \frac{t_k - u}{t_k - t_j} + \frac{u - t_j}{t_k - t_j} t_k \]

Arbitrary Knots
Problem

Observation

• Schaefer’s Algorithm does not Reduce to the Lane-Riesenfeld Algorithm when the Knots are Uniformly Spaced.

Questions

• Does there Exist a Global Knot Insertion Algorithm that Reduces to the Lane-Riesenfeld Algorithm for Uniform Knots?

• If Such an Algorithm Exists, is it Unique?
Lane-Riesenfeld Algorithm:
Cubic B-splines -- Arbitrary Knots

\[ v_k = t_{k+1} - \frac{(t_{k+1} - u_{k-1})(t_{k+1} - u_k)}{(t_{k+1} - t_k)} \]
\[ w_k = u_k - \frac{(u_k - t_{k-1})(t_k - u_{k-1})}{(t_k - t_{k-1})} \]
Lane-Riesenfeld Algorithm:
Cubic B-splines -- Arbitrary Knots

\[
\begin{align*}
&\text{u}_1 \text{t}_2 \text{u}_2 \\
&\quad \alpha_2 \\
&\quad \frac{1}{2} \quad 1 - \alpha_2 \\
&\quad \frac{1}{2} \quad 1 - \gamma_2 \\
&\quad t_1 t_2 \left( t_0 + 3 t_3 \right) / 4 \\
&\quad t_2 t_3 \text{v}_2 \\
&\quad \beta_2 \\
&\quad \frac{1}{2} \quad 1 - \beta_2 \\
&\quad \frac{1}{2} \quad 1 - \beta_3 \\
&\quad t_1 t_2 t_3 \\
&\quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \\
&\quad t_0 t_1 t_2 \\
\end{align*}
\]
**Parameters for Lane-Riesenfeld Algorithm**

\[
\alpha_k = \frac{4(t_{k+1} - u_{k-1})(t_{k+1} - u_k)}{(t_{k+1} - t_{k-2})(t_{k+1} - t_{k-1})}
\]

\[
\beta_k = \frac{2(u_k - t_{k-1})(u_{k-1} - t_{k-1})}{(t_{k+2} - t_{k-1})(t_{k+1} - t_{k-1})(1 - \alpha_k)}
\]

\[
\gamma_k = \frac{4(u_k - v_k)}{t_{k-1} + 3t_{k+2} - 4v_k}
\]

\[
v_k = t_{k-1} + \frac{\beta_k (t_{k+2} - t_{k-1})}{2}.
\]
Summary

*Global Knot Insertion Algorithms*

- Exist for Arbitrary Knot Sequences
- Are Easily Derived from Blossoming
Open Problems

1. Find a Global Knot Insertion Algorithm for Arbitrary Degree that
   Reduces to the Lane-Riesenfeld Algorithm for Uniform Knots.

2. Find the Simplest Knot Insertion Algorithm for Arbitrary Degree that
   Reduces to the Lane-Riesenfeld Algorithm for Uniform Knots.
   • How Complicated are the Labels Along the Edges?