Positive Gordon–Wixom Coordinates

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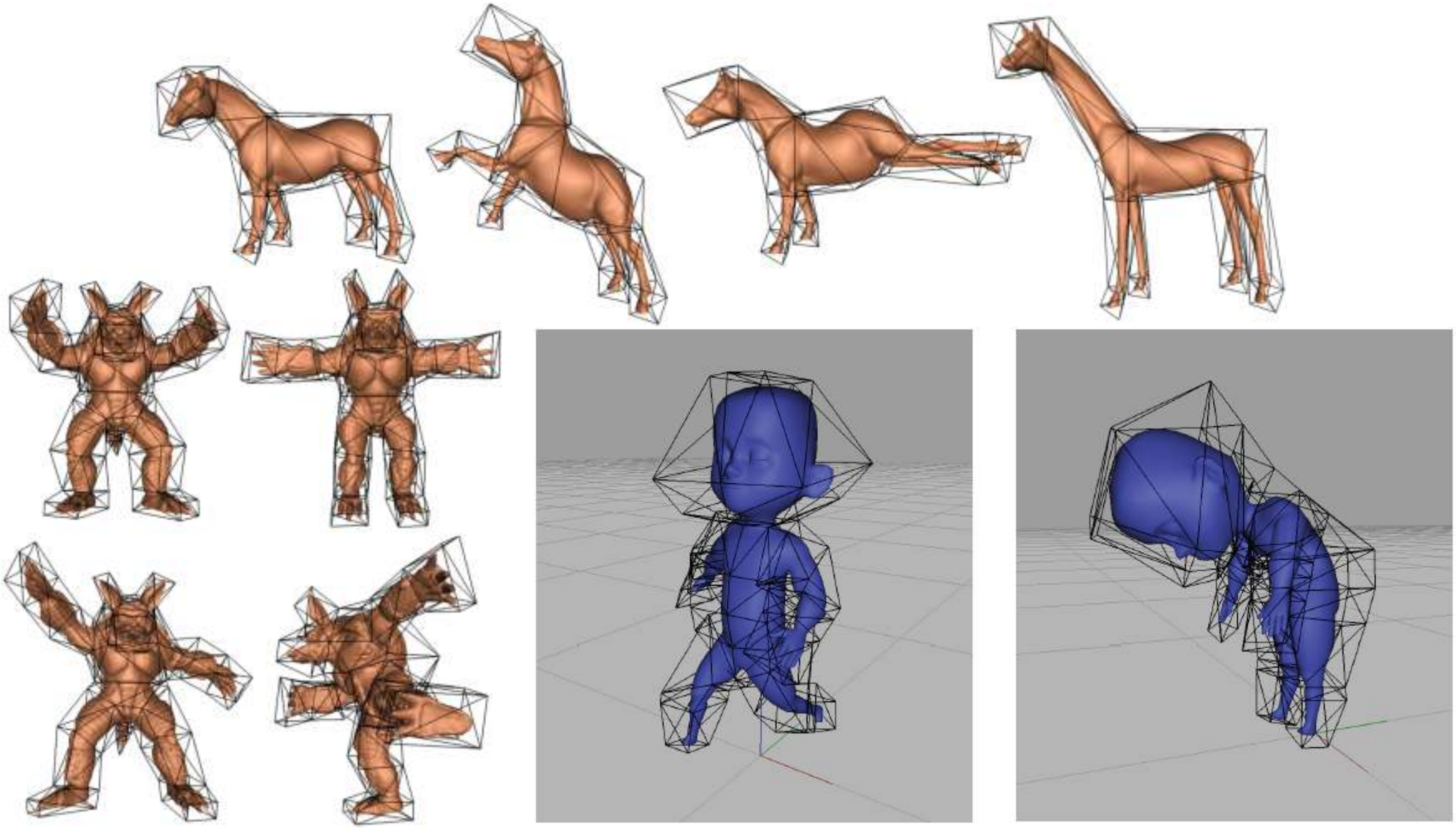
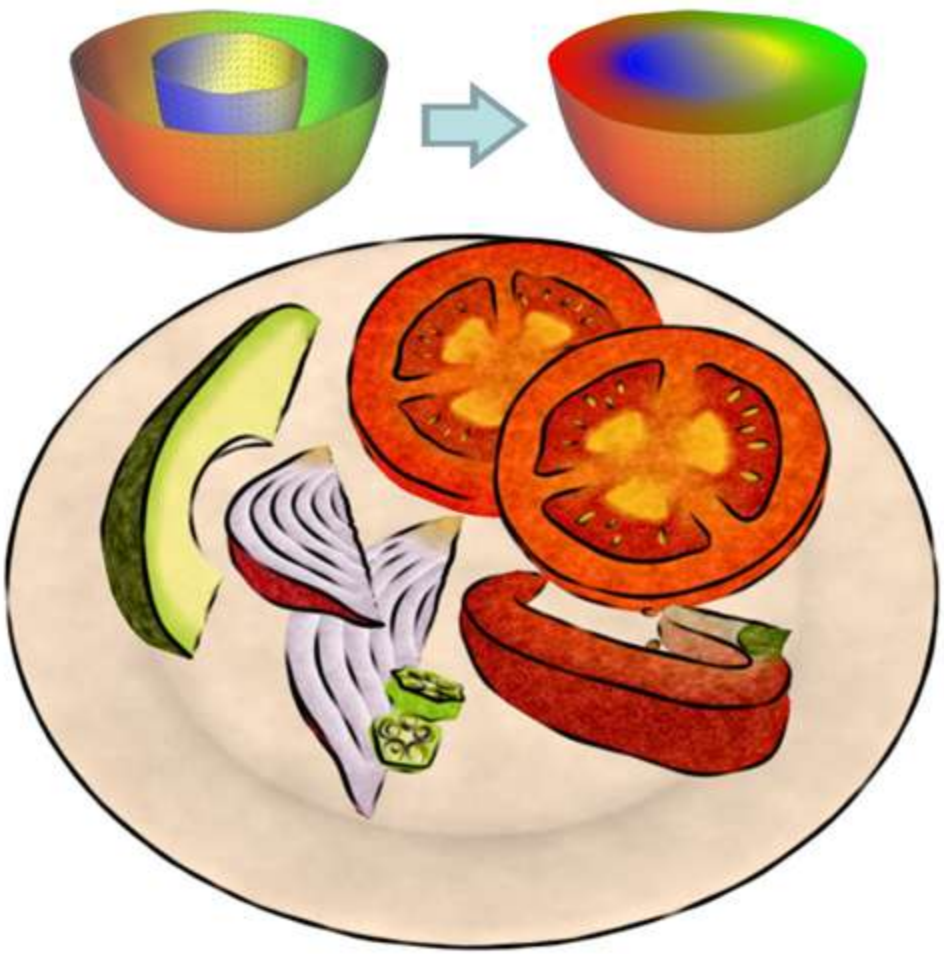
[2] Intel

Barycentric Coordinates: Mesh deformation



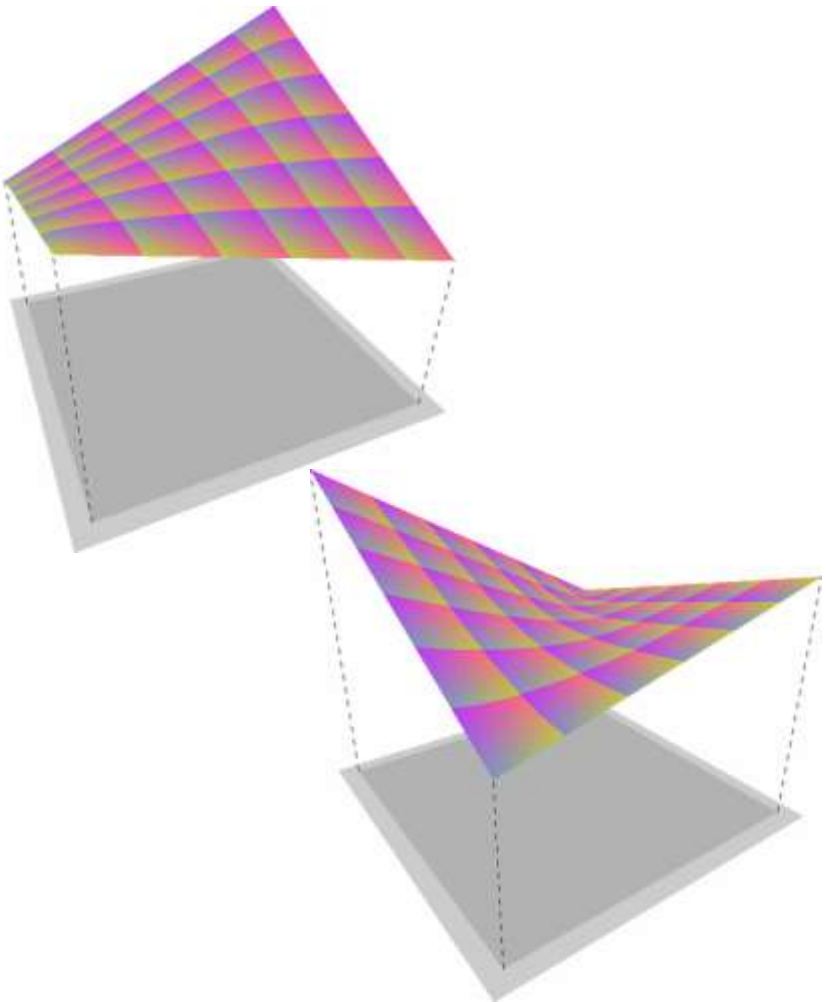
[Ju et al. 2005] [Joshi et al. 2007]

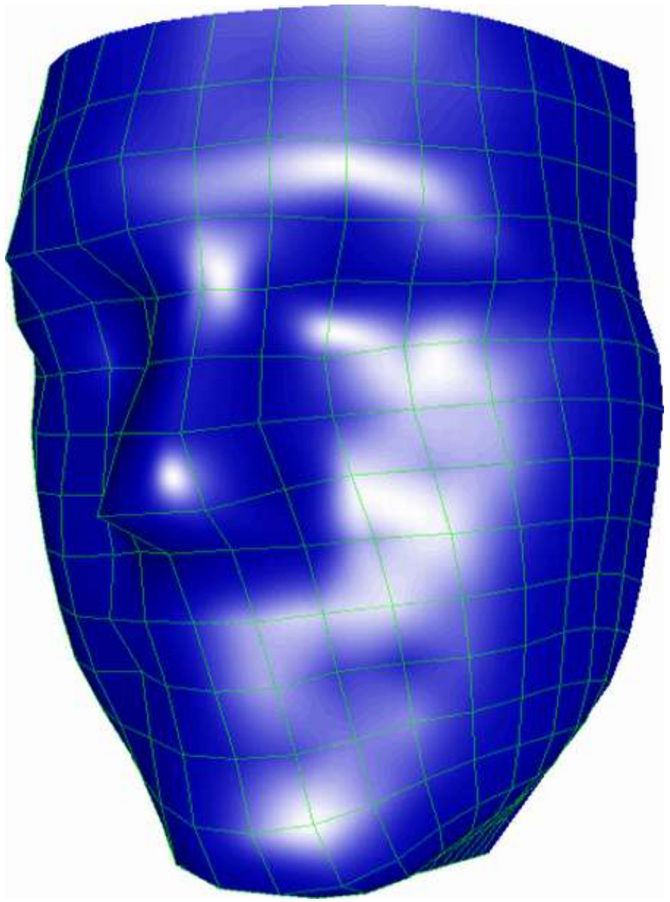
Barycentric Coordinates: Volumetric textures



[Takayama et al. 2010]

Barycentric Coordinates: Rasterize polygons



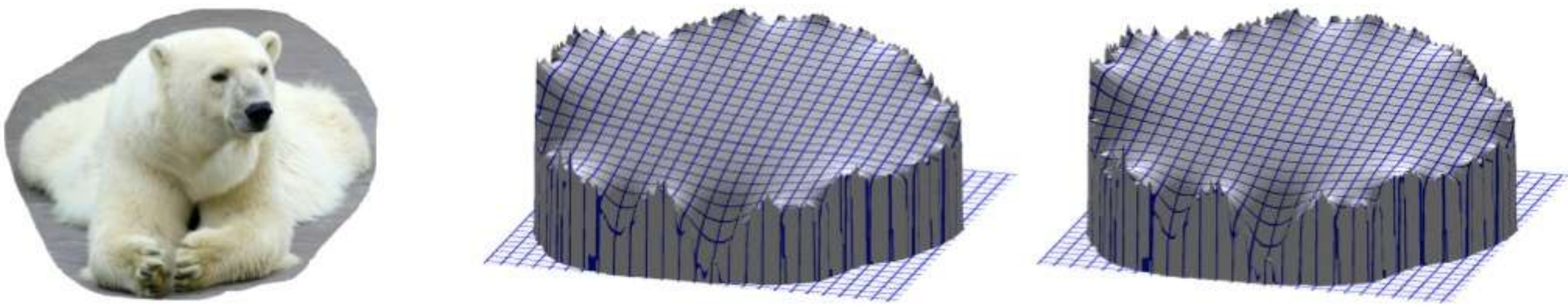


[Hormann and Tarini 2004]

Barycentric Coordinates: Surface representation

[Loop et al. 1989]

Barycentric Coordinates: Image editing



(a) Source patch (b) Laplace membrane (c) Mean-value membrane

(d) Target image (e) Poisson cloning (f) Mean-value cloning

[Farbman et al. 2009]

Barycentric Interpolant

$$f(x) = \int_{\partial\Omega} b(x,y) g(y)\ dy$$

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boundary values

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basis functions

coordinates

Basis Function Constraints

Boundary interpolation

$$x \in \partial\Omega \Rightarrow b(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

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Linear precision

$$\forall g \in \mathbb{P}^1, f(x) = g(x)$$

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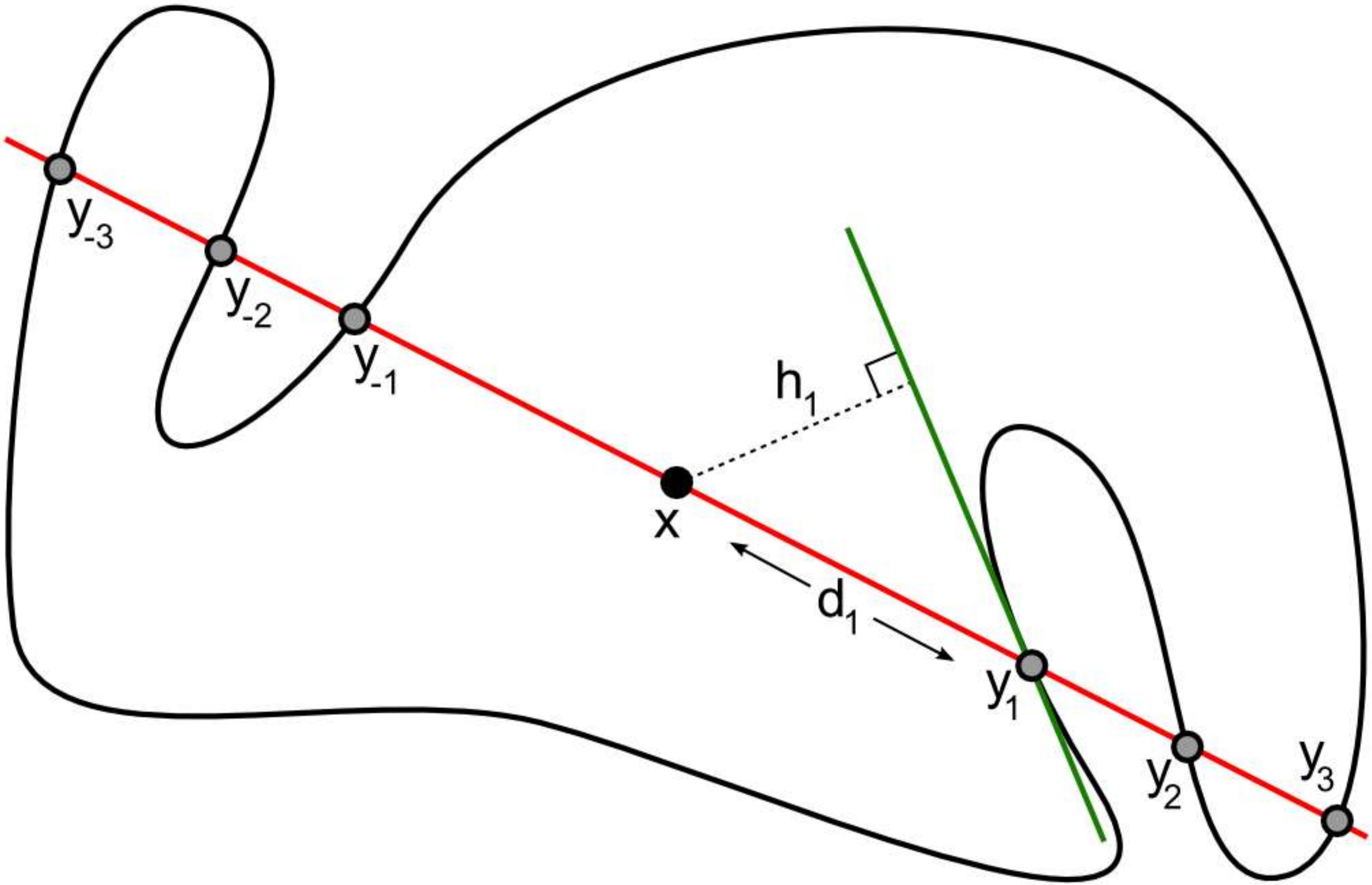
Positivity

$$b(x,y) \geq 0$$

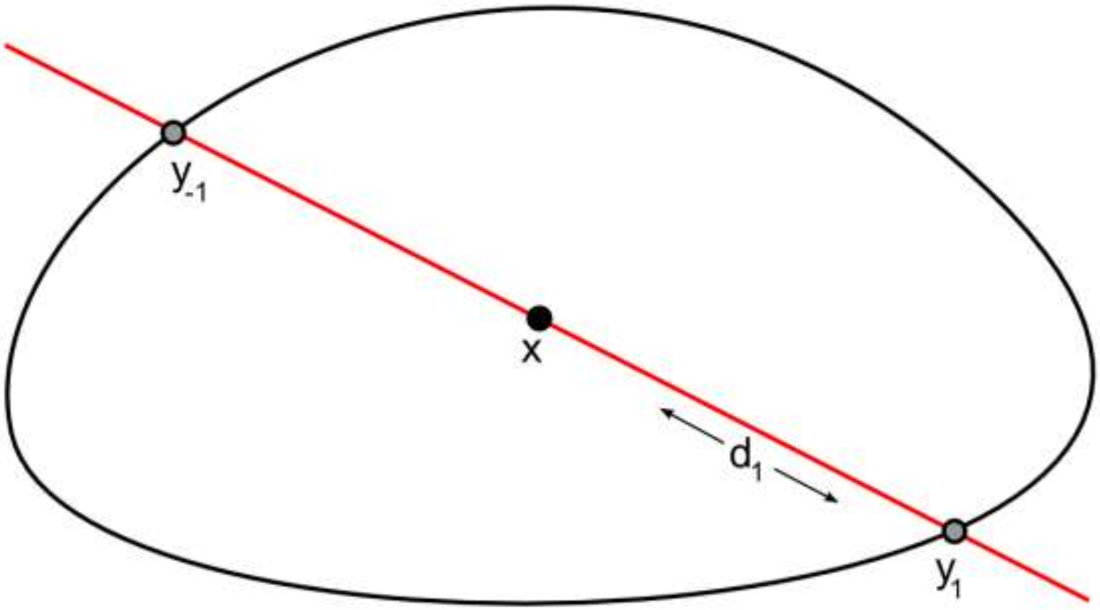
Types of Coordinates

- Wachspress [Wachspress 1975]
  - Only convex domains, obtuse angles bad
- Gordon-Wixom [Gordon and Wixom 1974]
  - Only convex domains
- Mean Value [Floater 2003]
  - Negative, but fast
- Moving Least Squares [Manson and Schaefer 2010]
  - Negative, but less so, slower
- Harmonic [Joshi et al. 2007]
  - Positive, ideal, very slow
- Maximum Entropy [Hormann and Sukumar 2008]
  - Positive, non-linear optimization, probably smooth
- Positive Gordon Wixom
  - Positive, evaluate integral, smooth for smooth boundaries

Notation

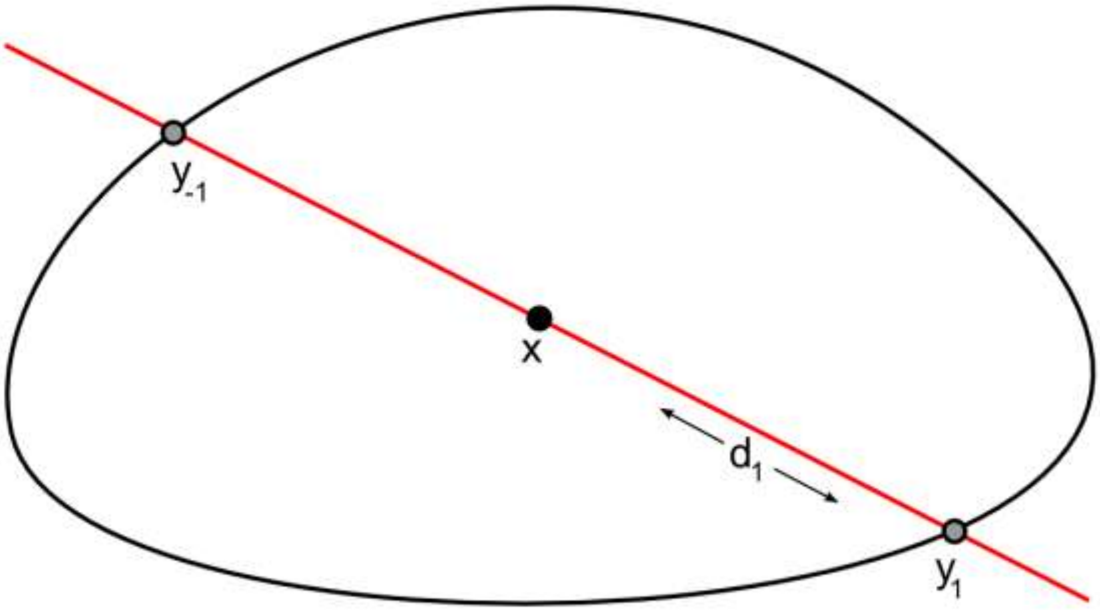


Linear Interpolant



$$L_{i,j}(x,\theta) = \frac{d_j}{d_i + d_j} g(y_i) + \frac{d_i}{d_i + d_j} g(y_j)$$

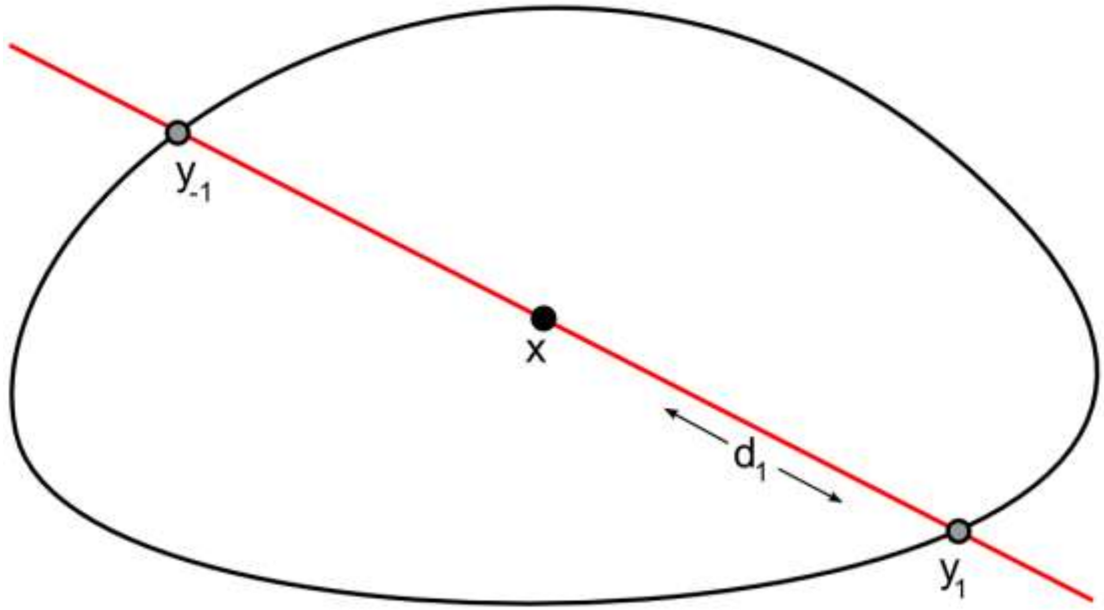
Gordon-Wixom



$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} L_{-1,1}(x, \theta)\ d\theta$$

[Gordon and Wixom 1974]

Weighted Gordon-Wixom



$$f(x) = \frac{\int_0^{2\pi} L_{-1,1}(x,\theta)\, W_{-1,1}(x,\theta)\; d\theta}{\int_0^{2\pi} W_{-1,1}(x,\theta)\; d\theta}$$

[Belyaev 2006]

Mean Value Coordinates (MVC)

$$W_{-1,1}(x,\theta) = \frac{d_1 + d_{-1}}{d_1 d_{-1}}$$

$$L_{i,j}(x,\theta) = \frac{d_j}{d_i + d_j} g(y_i) + \frac{d_i}{d_i + d_j} g(y_j)$$

$$f(x) = \frac{\int_0^{2\pi} \left(\frac{g(y_1)}{d_1} + \frac{g(y_{-1})}{d_{-1}}\right) d\theta}{\int_0^{2\pi} \left(\frac{1}{d_1} + \frac{1}{d_{-1}}\right) d\theta}$$

[Floater 2003]

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[Floater 2003]

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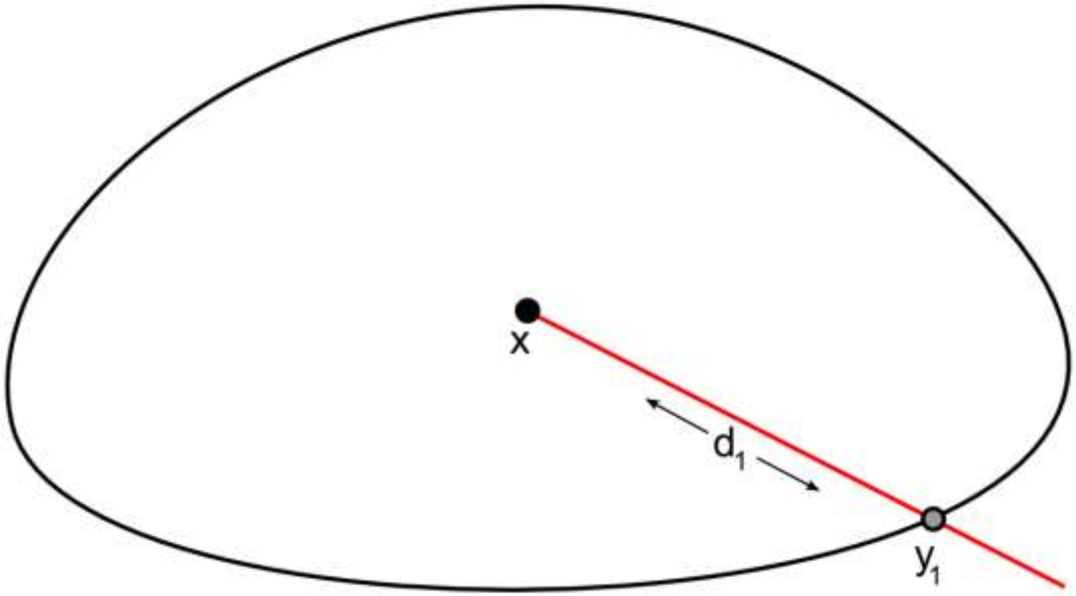

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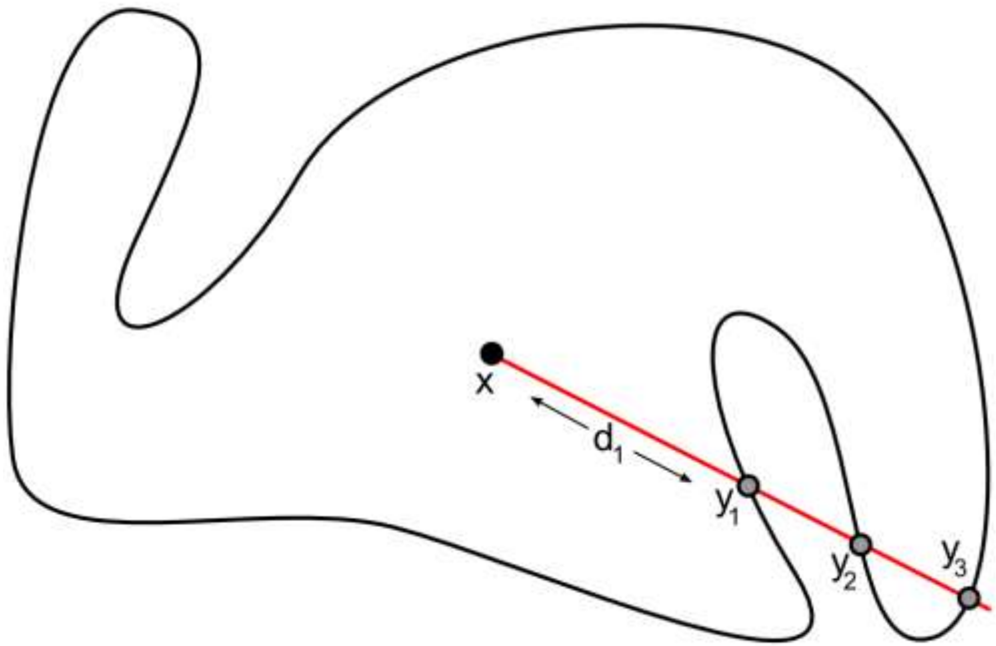
[Floater 2003]

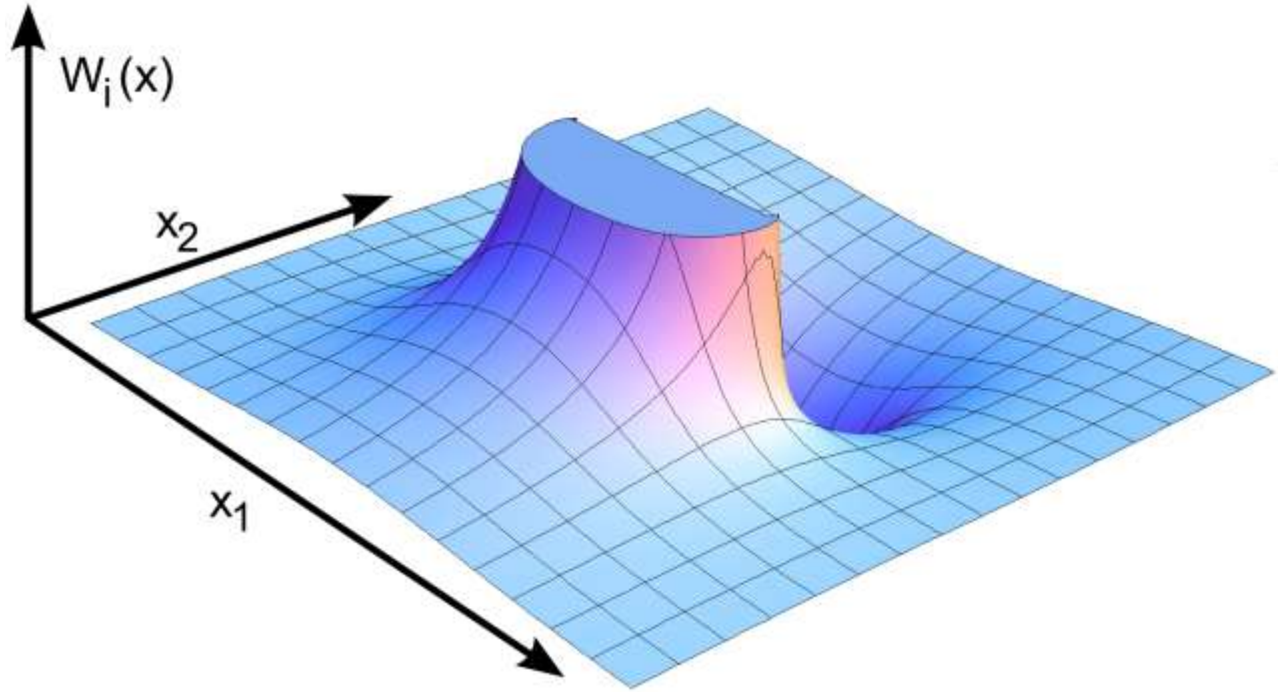
Concave MVC



$$f(x) = \frac{\int_0^{2\pi} \sum_{i=1}^{m} g(y_i) \frac{(-1)^{i+1}}{d_i} \, d\theta}{\int_0^{2\pi} \sum_{i=1}^{m} \frac{(-1)^{i+1}}{d_i} \, d\theta}$$

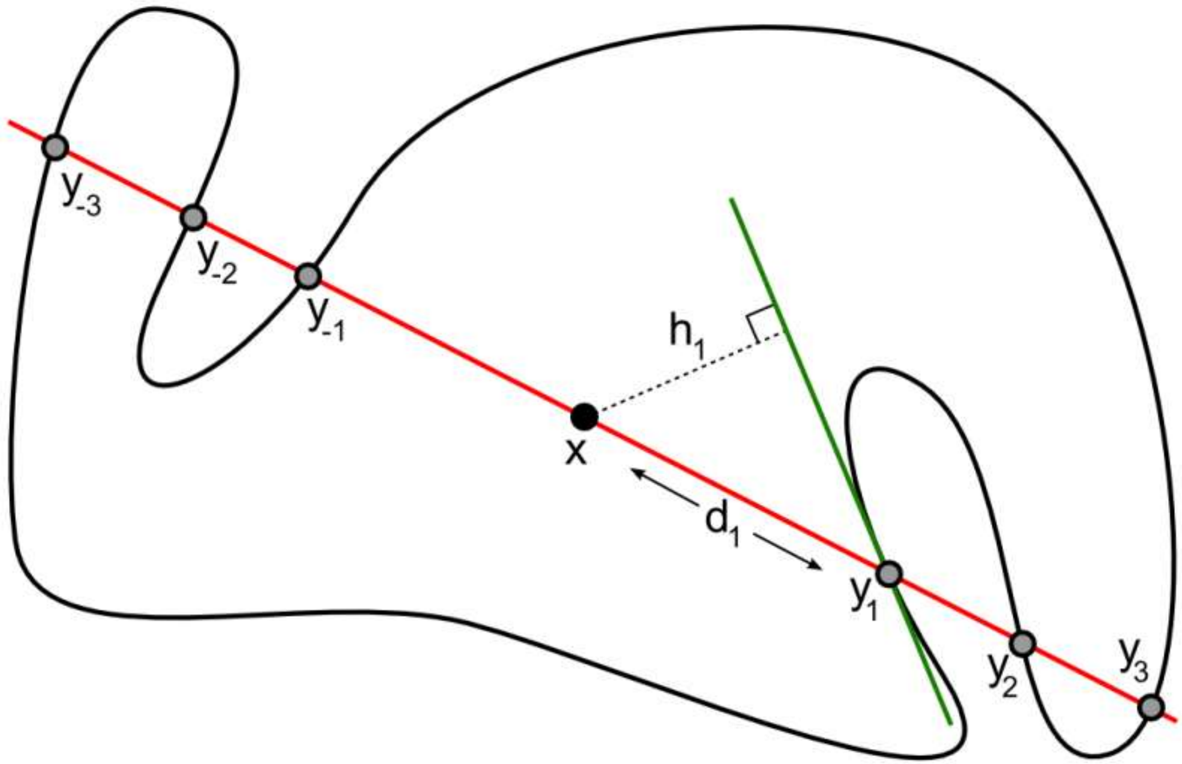
[Hormann and Floater 2006]

Concave MVC



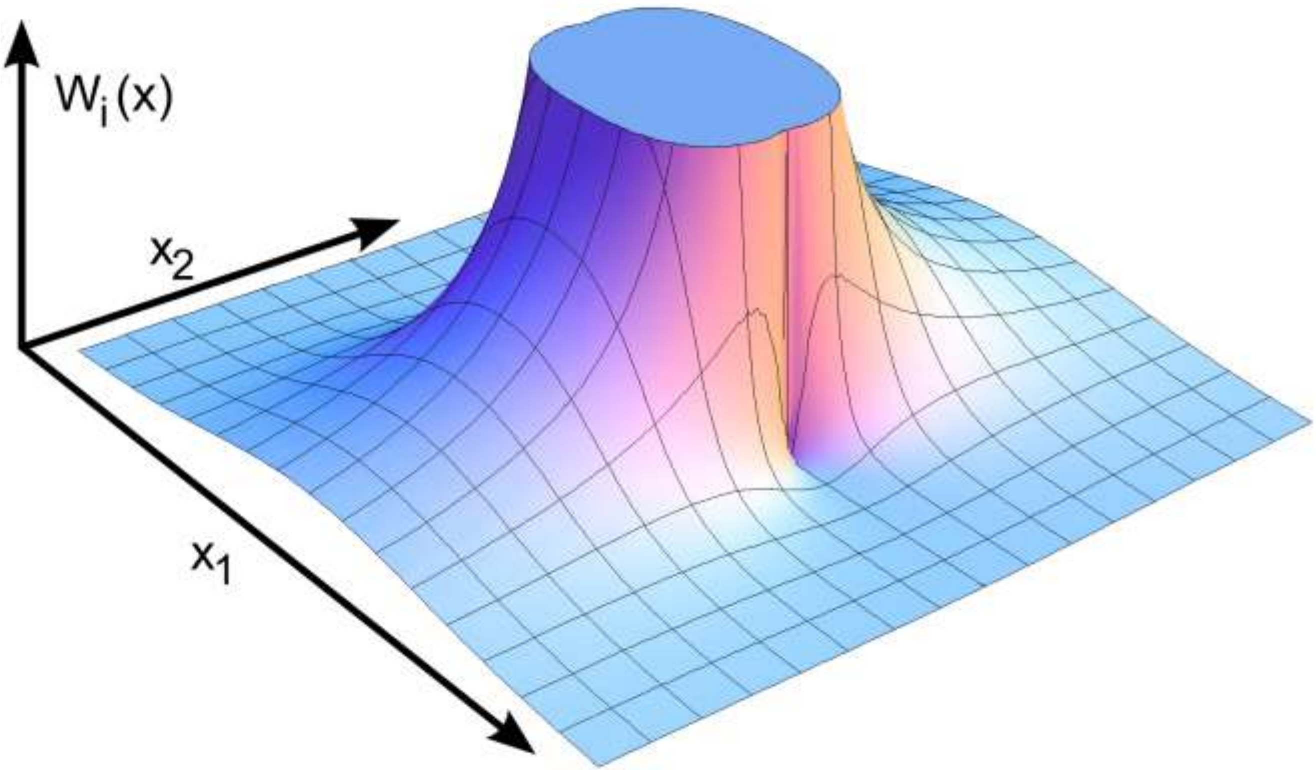
$$W_i(x, \theta) = \frac{1}{d_i}$$

Our Coordinates



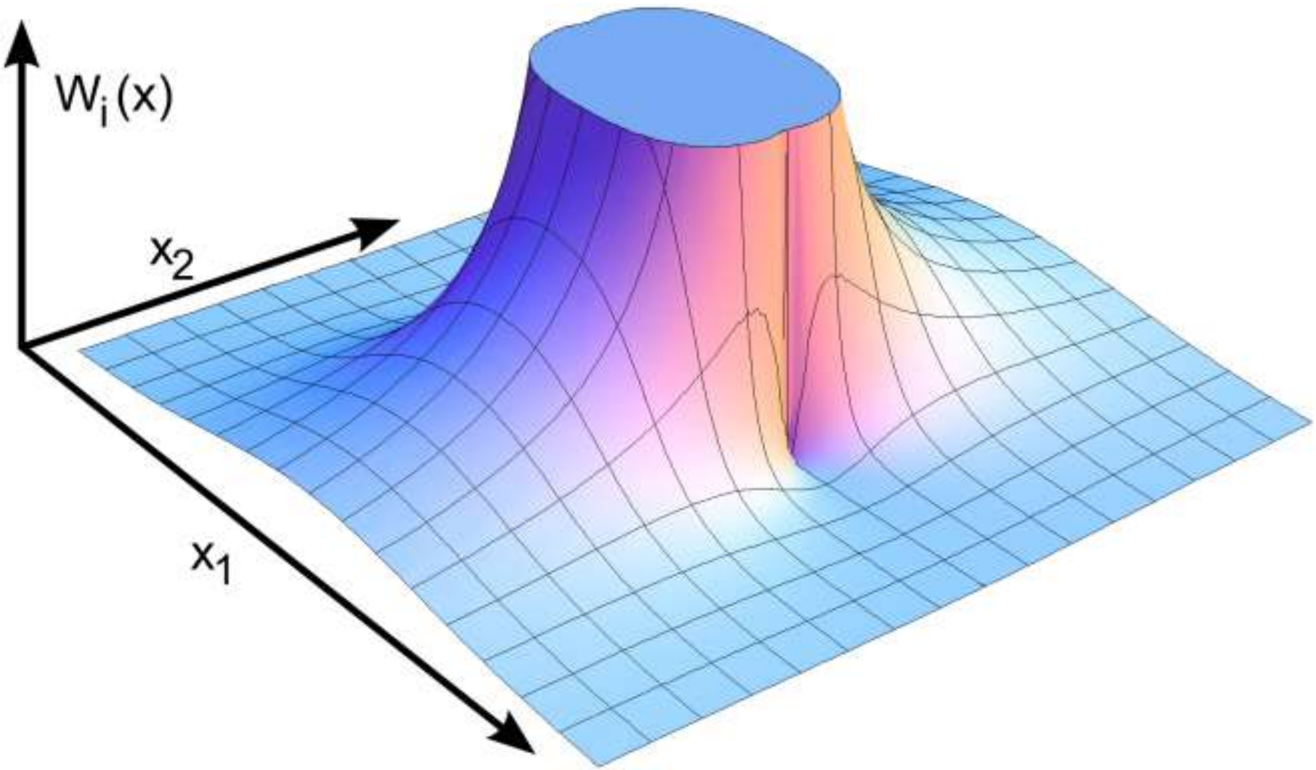
$$f(x) = \frac{\int_0^{2\pi} \sum_{i=1}^{m} \sum_{j=-n}^{-1} L_{i,j}(x,\theta) W_{i,j}(x,\theta)\ d\theta}{\int_0^{2\pi} \sum_{i=1}^{m} \sum_{j=-n}^{-1} W_{i,j}(x,\theta)\ d\theta}$$

Our Weight Function



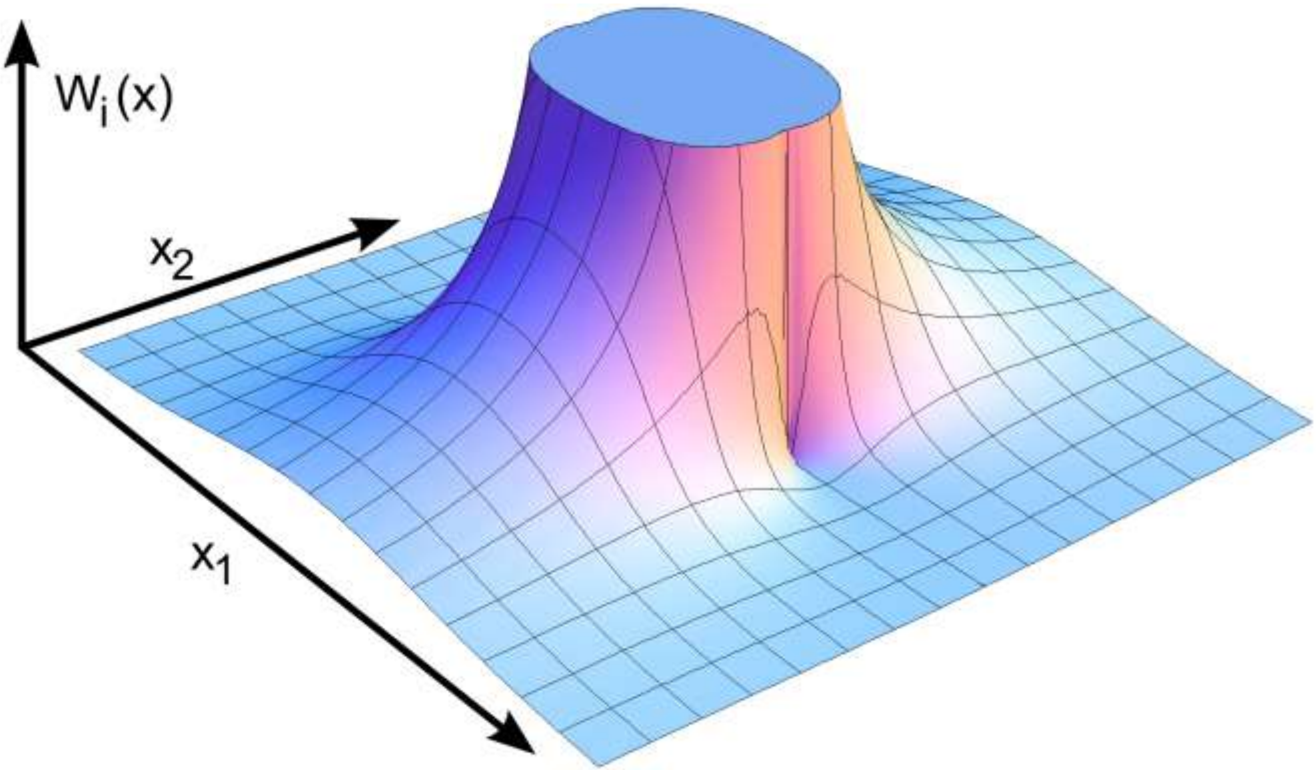
$$W_i(x, \theta) = \frac{h_i}{d_i^2}$$

Our Weight Function



$$W_{i,j}(x,\theta) = (d_i + d_j)W_i(x,\theta)W_j(x,\theta)$$

Our Weight Function



$$W_{i,j}(x, \theta) = \frac{(d_i + d_j)h_i h_j}{d_i^2 d_j^2}$$

Basis Functions

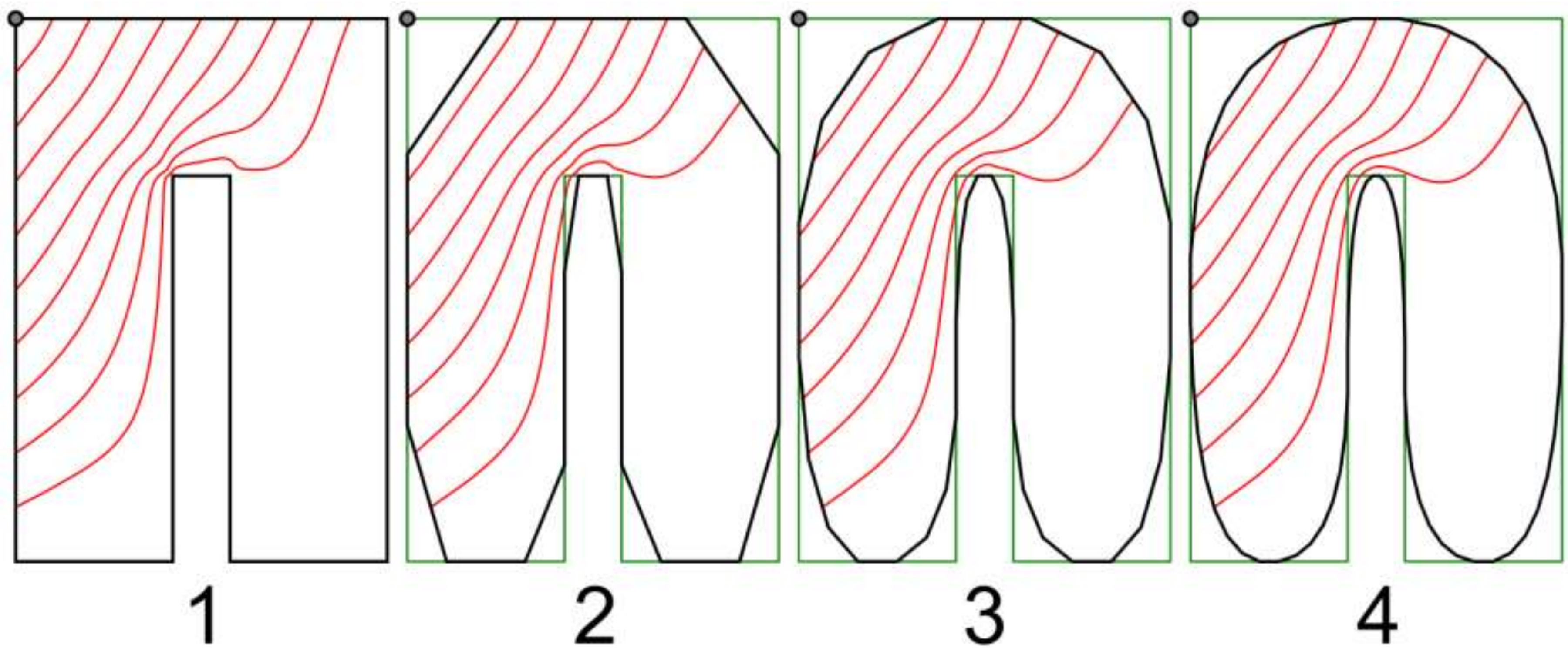
$$f(x) = \frac{\int_0^{2\pi} \sum_{i=1}^{m} \sum_{j=-n}^{-1} L_{i,j}(x,\theta) W_{i,j}(x,\theta)\; d\theta}{\int_0^{2\pi} \sum_{i=1}^{m} \sum_{j=-n}^{-1} W_{i,j}(x,\theta)\; d\theta}$$

$$W_{i,j}(x,\theta) = \frac{(d_i + d_j) h_i h_j}{d_i^2 d_j^2}$$

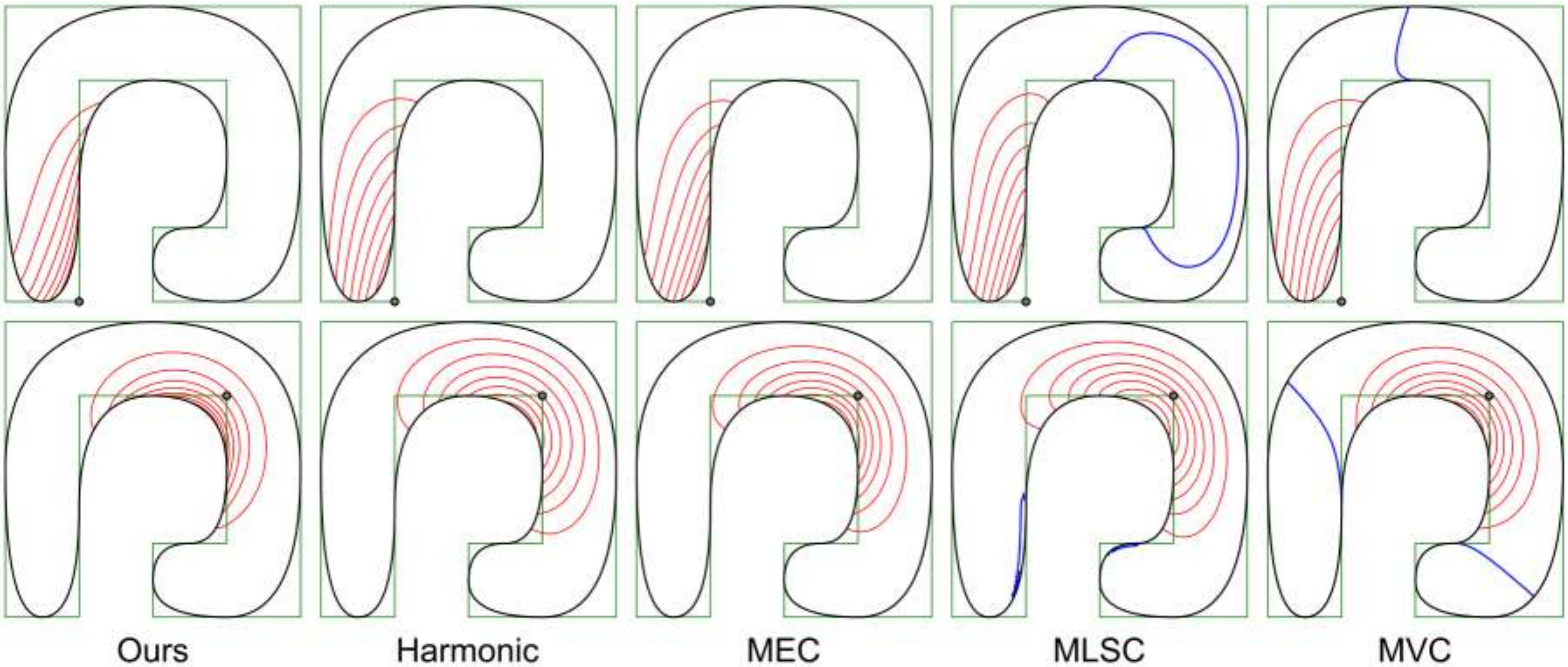
↓

$$b(x, y_i) = \frac{\sum_{j=-n}^{-1} \frac{h_i h_j}{d_i^2 d_j}}{\int_0^{2\pi} \sum_{i=1}^{m} \sum_{j=-n}^{-1} W_{i,j}(x,\theta)\; d\theta}$$

Approximating Smooth Boundaries



Comparison



Conclusion

- Our coordinates are:
  - Positive
  - Smooth for smooth boundary
  - Evaluated through integral
  - Closed-form for polygons
- Need visibility through sample point
  - Logarithmic lookup
  - Slows computation
- Evidence that closed-form for polygons exists