1. How many ways are there to pick \( r \) objects from \( n \) objects when repetitions are allowed and not all the \( n \) objects appear at the same time?

**Solution:**

(a) pick \( r \) objects from \( n \) objects when repetitions are allowed: \( \binom{n+r-1}{r} \)

(b) pick \( r \) objects from \( n \) objects when repetitions are allowed and ALL the \( n \) objects appear, which equals to first pick all \( n \) objects from \( n \) objects and then pick \( r-n \) objects from \( n \) objects (repetitions allowed): \( \binom{n+(r-n)-1}{r-n} = \binom{r-1}{r-n} \)

(c) pick \( r \) objects from \( n \) objects when repetitions are allowed and NOT ALL the \( n \) objects appear: \( \binom{n+r-1}{r} - \binom{r-1}{r-n} \)

2. How many permutations can be formed from \( n \) types of indistinguishable objects with objects of type \( i \) appearing \( i \) times for \( 1 \leq i \leq n \)?

**Solution:**

\[
\begin{align*}
N_1 &= 1 \text{ objects for type 1} \\
N_2 &= 2 \text{ objects for type 2} \\
&\vdots \\
N_n &= n \text{ objects for type } n \\
\text{Totally } N \text{ objects, } N &= N_1 + N_2 + \ldots + N_n = 1 + 2 + \ldots + n = n(n+1)/2 \\
\text{Totally permutations } \frac{N!}{N_1!N_2!\ldots N_n!} &= \frac{(n(n+1)/2)!}{1!2!\ldots n!}
\end{align*}
\]

3. Find an explicit formula for the recurrence relation \( a_n = (\sqrt{3} + \sqrt{2})a_{n-1} - \sqrt{6}a_{n-2} \) with initial conditions \( a_0 = 2 \) and \( a_1 = \sqrt{3} + \sqrt{2} \).

**Solution:**

\[
\begin{align*}
r^2 - (\sqrt{3} + \sqrt{2})r + \sqrt{6} &= 0 \\
r &= \sqrt{3} \text{ or } r = \sqrt{2} \\
a_n &= \alpha_1(\sqrt{3})^n + \alpha_2(\sqrt{2})^n \\
a_0 &= 2 \text{ and } a_1 = \sqrt{3} + \sqrt{2}, \text{ so } \alpha_1 = 1, \alpha_2 = 1 \\
a_n &= (\sqrt{3})^n + (\sqrt{2})^n
\end{align*}
\]

4. Solve the recurrence relation \( T(n) = 2T(n/3) + \log(2^n) \), \( T(1) = 1 \), by finding an expression for \( T(n) \) in big-Oh notation.

**Solution:**

\[
\begin{align*}
T(n) &= 2T(n/3) + c \log(2) \cdot n \\
a &= 2, b = 3, d = 1, a < b^d \\
T(n) &= O(n^1) = O(n)
\end{align*}
\]
5. Find the generating function to determine the number of ways to choose $k$ objects from $n$ objects when the $i$th object appears either $i - 1$, $i$ or $i + 1$ times for $1 \leq i \leq n$.

**Solution:**

generating function for the $i^{th}$ object: $X^{i-1}, X^i, X^{i+1}$

$$\prod_{i=1}^{n}(X^{i-1}, X^i, X^{i+1}) = (1 + X + X^2) \prod_{i=1}^{n} X^{i-1} = (1 + X + X^2) \prod_{i=0}^{n-1} X^i$$

$$= (1 + X + X^2)X^{n(n-1)/2}$$