Question 1 (10 pt):

\[
\sum_{k=1}^{n} \log \left(1 + \frac{2}{k}\right) = \sum_{k=1}^{n} \log \left(\frac{k+2}{k}\right) = \sum_{k=1}^{n} \log (k+2) - \log k \\
= \log (n+2) - \log n + \log (n+1) - \log (n+1) \\
+ \ldots \\
+ \log 3 - \log 2 + (\log 3) - \log 1 \\
= \log \frac{(n+2)(n+1)}{2}
\]

Question 2 (10 pt):

Solution 1:

let \( f(k) = \frac{1}{k} \) and \( g(k) = 2^k \), then \( \Delta f(k) = -\frac{1}{k(k+1)} \) and \( \Delta g(k) = 2^k \). So:

\[
\sum_{k=1}^{n} \left(\frac{2^k}{k} - \frac{2^{k+1}}{k(k+1)}\right) = \sum_{k=1}^{n} g(k+1) \Delta f(k) + f(k) \Delta g(k) \\
= \sum_{k=1}^{n} \Delta (f(k)g(k)) \\
= f(n+1)g(n+1) - f(1)g(1) \\
= \frac{2^{n+1}}{n+1} - 2
\]

Solution 2:

\[
\sum_{k=1}^{n} \left(\frac{2^k}{k} - \frac{2^{k+1}}{k(k+1)}\right) = \sum_{k=1}^{n} \left(\frac{2^k}{k} - \left(\frac{2^{k+1}}{k} - \frac{2^{k+1}}{k+1}\right)\right) \\
= \sum_{k=1}^{n} \left(\frac{2^{k+1}}{k+1} - \frac{2^k}{k}\right) \\
= \frac{2^{n+1}}{n+1} - 2
\]
Question 3 (10 pt):

Solution 1:

According to $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + o\left(\frac{1}{n}\right)\right) \rightarrow n! \simeq \left(\frac{n}{e}\right)^n$, we can get

$(n^2)! \simeq \left(\frac{n^2}{e}\right)^{n^2}$ and $(n!)^n \simeq \left(\left(\frac{n}{e}\right)^n\right)^n$. Then we can get

$$\lim_{n \to \infty} \frac{(n^2)!}{(n!)^n} \simeq \frac{\left(\frac{n^2}{e}\right)^{n^2}}{\left(\left(\frac{n}{e}\right)^n\right)^n}$$

$$= \frac{\left(\frac{n^2}{e}\right)^{n^2}}{\left(\frac{n}{e}\right)^{n^2}} = n^{n^2} = \infty$$

So, we can conclude that $f(n) = o(g(n))$
Question 4 (10 pt):

\[ f(n) = 2f(n/2) + n \]
\[ = 2(2(f(n/4) + n/2)) + n \]
\[ = ... \]
\[ = 2^\log_2 n f(1) + n \log_2 n \]
\[ = n + n \log n \]

Question 5 (10 pt):

Answer 1:
The leaf of the decision tree is \( n \), the number of the comparsion need to be made is the high of the decision tree. Since \( 2^h > n \), we can then get \( h > \log_2 n \)

Answer 2:
If it is an unsorted array, you need to compare all the number, so the lower-bound is \( n \).

Both Answer 1 and Answer 2 will be accepted.